CRYSTAL GRAPHS FOR LIE SUPERALGEBRAS AND CAUCHY DECOMPOSITION

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Abstract. We discuss Cauchy type decompositions of crystal graphs for general linear Lie superalgebras. More precisely, we consider bicrystal graph structures on various sets of matrices of non-negative integers, and obtain their decompositions with explicit combinatorial isomorphisms.

1. introduction

Let $\mathfrak{gl}_{m|n}$ be the general linear Lie superalgebra over $\mathbb{C}$, and let $\mathbb{C}^{m|n}$ be its $(m+n)$-dimensional natural representation with the $\mathbb{Z}_2$-grading $(\mathbb{C}^{m|n})_0 = \mathbb{C}^m$ and $(\mathbb{C}^{m|n})_1 = \mathbb{C}^n$. A tensor power of $\mathbb{C}^{m|n}$ is completely reducible from Schur-Weyl duality, and its irreducible components, called irreducible polynomial representations, are parameterized by $\mathcal{P}_{m|n}$, the set of all (m,n)-hook partitions [2]. Let $S(\mathbb{C}^{m|n}\otimes\mathbb{C}^{u|v})$ be the (super) symmetric algebra generated by the $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$-bimodule $\mathbb{C}^{m|n}\otimes\mathbb{C}^{u|v}$. From Howe duality, it is also completely reducible as a $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$-bimodule, and we have the following Cauchy type decomposition;

$$S(\mathbb{C}^{m|n}\otimes\mathbb{C}^{u|v}) = \bigoplus_{\lambda\in\mathcal{P}_{m|n}\cap\mathcal{P}_{u|v}} V_{m|n}(\lambda)\otimes V_{u|v}(\lambda), \tag{1.1}$$

where $V_{m|n}(\lambda)$ and $V_{u|v}(\lambda)$ denote the irreducible polynomial representations of $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_{u|v}$, respectively, corresponding to λ (see [4, 6]). In terms of characters, (1.1) also yields a Cauchy type identity of hook Schur functions (cf.[15]).

The purpose of this paper is to understand the decomposition (1.1) within a framework of (abstract) crystal graphs for Lie superalgebras which were developed by Benkart, Kang and Kashiwara [1]. For $\lambda\in\mathcal{P}_{m|n}$, we denote by $\mathbf{B}_{m|n}(\lambda)$ the set of all (m,n)-hook semistandard tableaux of shape λ, which parameterizes the basis element of $V_{m|n}(\lambda)$ [2]. According to the crystal base theory in [1], $\mathbf{B}_{m|n}(\lambda)$ becomes a colored oriented graph, which we call a *crystal graph for* $\mathfrak{gl}_{m|n}$ or $\mathfrak{gl}_{m|n}$-*crystal*. As in the case of symmetrizable Kac-Moody algebras, the crystal graphs for $\mathfrak{gl}_{m|n}$ have nice behaviors under tensor product, and we can decompose various finite dimensional representations of $\mathfrak{gl}_{m|n}$ in a purely combinatorial way (cf.[11]).

For non-negative integers m,n,u,v such that $m+n, u+v>0$, let

$$\mathbf{M} = \{\, A=(a_{bb'})_{b\in\mathbf{B}_{m|n}, b'\in\mathbf{B}_{u|v}} \mid \text{(i) } a_{bb'}\in\mathbb{Z}_{\geq 0},\ \text{(ii) } a_{bb'}\leq 1 \text{ if } |b|\neq|b'|\,\},$$

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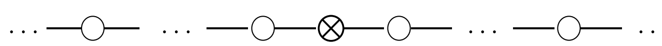
where $\mathbf{B}_{m|n}$ (resp.$\mathbf{B}_{u|v}$) is the crystal graph associated to the natural representation of $\mathfrak{gl}_{m|n}$ (resp. $\mathfrak{gl}_{u|v}$), and $|b|$ denotes the degree of b. Note that $\mathbf{M}$ naturally parameterizes the set of monomial basis of $S(\mathbb{C}^{m|n}\otimes\mathbb{C}^{u|v})$. Then we show that as a crystal graph for $\mathfrak{gl}_{m|n}\oplus\mathfrak{gl}_{u|v}$ (or $(\mathfrak{gl}_{m|n},\mathfrak{gl}_{u|v})$-bicrystal),

$$\mathbf{M}\simeq\bigoplus_{\lambda\in\mathcal{P}_{m|n}\cap\mathcal{P}_{u|v}}\mathbf{B}_{m|n}(\lambda)\times\mathbf{B}_{u|v}(\lambda). \tag{1.2}$$

The isomorphism is given by a super-analogue of the well-known Knuth correspondence [14]. But our proof is different from the original one since the decomposition is given by characterizing all the highest weight elements of the connected components from the view point of crystal graphs. Furthermore, our approach enables us to explain several variations of the Knuth correspondence (cf.[5]) in a unified way, and also derive an interesting relation between the statistics of the diagonal entries of a symmetric matrix in $\mathbf{M}$ and the number of odd parts in the shape of the corresponding tableau (a special case of this relation was first observed in [14]).

We may naturally extend the above decomposition to a semi-infinite case. Let $\mathfrak{g}$ be a contragredient Lie superalgebra of infinite rank whose Dynkin diagram is given by



(see [8]). First, we introduce a $\mathfrak{g}$-crystal $\mathscr{F}$ consisting of semi-infinite words, which can be viewed as a crystal graph associated to a Fock space representation of $\mathfrak{g}$ analogous to the level one fermionic Fock space representation of $\widehat{\mathfrak{gl}}_\infty$ (cf.[7]). We show that a connected component of a tensor power $\mathscr{F}^{\otimes u}$ ($u\geq 1$) can be realized as the set of semi-infinite semistandard tableaux, which is generated by a highest weight element. Moreover, an explicit multiplicity-free decomposition of $\mathscr{F}^{\otimes u}$ as a $(\mathfrak{g},\mathfrak{gl}_u)$-bicrystal is given, where each connected component is parameterized by a generalized partition of length u. To prove this, we identify $\mathscr{F}^{\otimes u}$ with a set of certain matrices with infinite number of rows, and apply the methods used in the case of finite ranks. More precisely, an element in $\mathscr{F}^{\otimes u}$ is equivalent to a unique pair of an semi-infinite semistandard tableau and a rational semistandard tableau as an element in a $(\mathfrak{g},\mathfrak{gl}_u)$-bicrystal. Hence, it gives rise to a Knuth correspondence of a semi-infinite type, and a Cauchy type identity. As a by-product, we obtain a character formula of a $\mathfrak{g}$-crystal of semi-infinite semistandard tableaux occurring as a connected component in $\mathscr{F}^{\otimes u}$. This character formula is given in terms of ordinary Schur functions and the Littlewood-Richardson coefficients, and it is very similar to the ones of the irreducible highest weight representations of $\widehat{\mathfrak{gl}}_\infty$ or $\widehat{\mathfrak{gl}}_{\infty|\infty}$ obtained in [3, 10]. In fact, using crystal graphs, we can give a similar combinatorial proof of the Cauchy type decomposition of a higher level Fock space representation of $\widehat{\mathfrak{gl}}_\infty$ given by Kac and Radul [10], and hence the character formula of irreducible highest weight representations of $\widehat{\mathfrak{gl}}_\infty$. We also expect a combinatorial proof of the decomposition of a Fock space representation of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ given by Cheng and Lam [3].

The paper is organized as follows. In Section 2, we review the basic notions and the main results on crystal graphs for $\mathfrak{gl}_{m|n}$ in [1]. In Section 3, we prove the decomposition (1.2) and then study the diagonal action of the Kashiwara operators on the set of symmetric matrices in $\mathbf{M}$. In Section 4, we describe the dual decomposition which is associated to the (super) exterior algebra $\Lambda(\mathbb{C}^{m|n}\otimes\mathbb{C}^{u|v})$. Finally, in Section 5, we introduce a $\mathfrak{g}$-crystal of semi-infinite semistandard tableaux, and a $\mathfrak{gl}_u$-crystal of rational semistandard tableaux (cf.[17]) . Then using these combinatorial realizations of crystal graphs, we establish a Cauchy type decomposition of a tensor power $\mathscr{F}^{\otimes u}$ as a $(\mathfrak{g},\mathfrak{gl}_u)$-bicrystal.

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2. Crystal graphs for $\mathfrak{gl}_{m|n}$

In this section, we recall the basic notions on crystal graphs for $\mathfrak{gl}_{m|n}$ developed in [1].

2.1. **Definitions.** For non-negative integers m,n with $m+n>0$, let $\mathfrak{gl}_{m|n}$ be the general linear Lie superalgebra over $\mathbb{C}$ (see [8]). Let

$$\mathbf{B}_{m|n}=\{\,\overline{m}<\overline{m-1}<\cdots<\overline{1}<1<2<\cdots<n\,\}$$

be a linearly ordered set. Set $\mathbf{B}^{+}_{m|n}=\{\,\overline{m},\overline{m-1},\cdots,\overline{1}\,\}$ and $\mathbf{B}^{-}_{m|n}=\{\,1,2,\cdots,n\,\}$. For $b\in\mathbf{B}_{m|n}$, we define $|b|$, *degree of* b, by $|b|=0$ (resp. 1) if $b\in\mathbf{B}^{+}_{m|n}$ (resp. $\mathbf{B}^{-}_{m|n}$). The free abelian group $P_{m|n}=\bigoplus_{b\in\mathbf{B}_{m|n}}\mathbb{Z}\epsilon_b$, which is generated by ϵ_b ($b\in\mathbf{B}_{m|n}$), is called the *weight lattice* of $\mathfrak{gl}_{m|n}$. There is a natural symmetric $\mathbb{Z}$-bilinear form $(\ ,\ )$ on $P_{m|n}$, where $(\epsilon_b,\epsilon_{b'})=(-1)^{|b|}\delta_{bb'}$ for $b,b'\in\mathbf{B}_{m|n}$. Let

$$I_{m|n}=\{\,\overline{m-1},\cdots,\overline{1},0,1,\cdots,n-1\,\}.$$

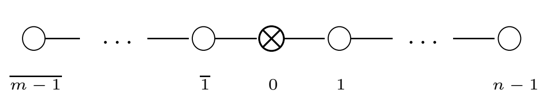
The *simple root* α_i ($i\in I_{m|n}$) of $\mathfrak{gl}_{m|n}$ is given by

$$\begin{cases}\alpha_{\overline{k}}=\epsilon_{\overline{k+1}}-\epsilon_{\overline{k}}, & \text{for } k=1,\cdots,m-1,\\ \alpha_l=\epsilon_l-\epsilon_{l+1}, & \text{for } l=1,\cdots,n-1,\\ \alpha_0=\epsilon_{\overline{1}}-\epsilon_1. & \end{cases}$$

Set $Q=\bigoplus_{i\in I_{m|n}}\mathbb{Z}\alpha_i$, which we call the *root lattice of* $\mathfrak{gl}_{m|n}$. A partial ordering on $P_{m|n}$ is given by $\lambda\geq\mu$ if and only if $\lambda-\mu\in\sum_{i\in I_{m|n}}\mathbb{Z}_{\geq 0}\alpha_i$ for $\lambda,\mu\in P_{m|n}$. We also define the *simple coroot* $h_i\in P^{*}_{m|n}$ ($i\in I_{m|n}$) by

$$\langle h_i,\lambda\rangle=\begin{cases}(\alpha_i,\lambda), & \text{if } i=\overline{m-1},\cdots,\overline{1},0,\\ -(\alpha_i,\lambda), & \text{if } i=1,\cdots,n-1,\end{cases}$$

for $\lambda\in P_{m|n}$, where $\langle\ ,\ \rangle$ is the natural pairing on $P^{*}_{m|n}\times P_{m|n}$. With respect to the above simple roots, the Dynkin diagram is



Motivated by the crystal bases of integral representations of the quantized enveloping algebra $U_q(\mathfrak{gl}_{m|n})$, we introduce the notion of abstract crystal graphs for $\mathfrak{gl}_{m|n}$.

Definition 2.1. (1) A *crystal graph for* $\mathfrak{gl}_{m|n}$ (or $\mathfrak{gl}_{m|n}$*-crystal*) is a set B together with the maps

$$\begin{aligned}&\mathrm{wt} : B \to P_{m|n},\\ &\varepsilon_i, \varphi_i : B \to \mathbb{Z}_{\geq 0},\\ &e_i, f_i : B \to B \cup \{0\},\end{aligned}$$

for $i \in I_{m|n}$ (0 is a formal symbol), satisfying the following conditions;

(a) for $i \in I_{m|n}$ and $b \in B$, we have

$$\varphi_i(b) - \varepsilon_i(b) = \langle h_i, \mathrm{wt}(b)\rangle, \quad (i \neq 0),$$

$$\langle h_0, \mathrm{wt}(b)\rangle \geq 0, \text{ and } \ \varphi_0(b) + \varepsilon_0(b) = \begin{cases} 0, & \text{if } \langle h_0, \mathrm{wt}(b)\rangle = 0,\\ 1, & \text{if } \langle h_0, \mathrm{wt}(b)\rangle > 0,\end{cases}$$

(b) if $e_i b \in B$ for $i \in I_{m|n}$ and $b \in B$, then

$$\varepsilon_i(e_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(e_i b) = \varphi_i(b) + 1, \quad \mathrm{wt}(e_i b) = \mathrm{wt}(b) + \alpha_i,$$

(c) if $f_i b \in B$ for $i \in I_{m|n}$ and $b \in B$, then

$$\varepsilon_i(f_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(f_i b) = \varphi_i(b) - 1, \quad \mathrm{wt}(f_i b) = \mathrm{wt}(b) - \alpha_i,$$

(d) $f_i b = b'$ if and only if $b = e_i b'$ for all $i \in I_{m|n}$, $b, b' \in B$,

(We call e_i and f_i ($i \in I_{m|n}$) the *Kashiwara operators*).

(2) Let B be a crystal graph for $\mathfrak{gl}_{m|n}$. A subset $B' \subset B$ is called a *subcrystal of* B if B' is itself a crystal graph for $\mathfrak{gl}_{m|n}$ with respect to $\mathrm{wt}, \varepsilon_i, \varphi_i, e_i, f_i$ ($i \in I_{m|n}$) of B.

Remark 2.2. (1) The above definition is based on the crystal bases of integrable representations of $U_q(\mathfrak{gl}_{m|n})$ in [1], while crystal graphs for contragredient Lie superalgebras might be defined in a more general sense, as in the case of symmetrizable Kac-Moody algebras (cf. [12, 13]).

(2) A crystal graph B becomes an $I_{m|n}$-colored oriented graph, where

$$b \xrightarrow{i} b' \quad \text{if and only if} \quad b' = f_i b \qquad (i \in I_{m|n}).$$

Definition 2.3. Let B_1 and B_2 be crystal graphs for $\mathfrak{gl}_{m|n}$. We define the *tensor product of* B_1 and B_2 to be the set $B_1 \otimes B_2 = \{\, b_1 \otimes b_2 \,|\, b_i \in B_i,\ (i = 1, 2)\,\}$ with

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),$$

$$\varepsilon_i(b_1 \otimes b_2) = \begin{cases} \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle), & \text{if } i = \overline{m-1}, \cdots, \overline{1},\\ \max(\varepsilon_i(b_1) - \langle h_i, \mathrm{wt}(b_2)\rangle, \varepsilon_i(b_2)), & \text{if } i = 1, \cdots, n-1,\\ \varepsilon_0(b_1), & \text{if } \langle h_0, \mathrm{wt}(b_1)\rangle > 0,\\ \varepsilon_0(b_2), & \text{if } \langle h_0, \mathrm{wt}(b_1)\rangle = 0,\end{cases}$$

$$\varphi_i(b_1\otimes b_2)=\begin{cases}\max(\varphi_i(b_1)+\langle h_i,\mathrm{wt}(b_2)\rangle,\varphi_i(b_2)), & \text{if } i=\overline{m-1},\cdots,\overline{1},\\ \max(\varphi_i(b_1),\varphi_i(b_2)+\langle h_i,\mathrm{wt}(b_1)\rangle), & \text{if } i=1,\cdots,n-1,\\ \varphi_0(b_1), & \text{if } \langle h_0,\mathrm{wt}(b_1)\rangle>0,\\ \varphi_0(b_2), & \text{if } \langle h_0,\mathrm{wt}(b_1)\rangle=0,\end{cases}$$

$$e_i(b_1\otimes b_2)=\begin{cases} e_ib_1\otimes b_2, & \text{if } i=\overline{m-1},\cdots,\overline{1},\ \varphi_i(b_1)\geq\varepsilon_i(b_2),\\ b_1\otimes e_ib_2, & \text{if } i=\overline{m-1},\cdots,\overline{1},\ \varphi_i(b_1)<\varepsilon_i(b_2),\\ e_ib_1\otimes b_2, & \text{if } i=1,\cdots,n-1,\ \varphi_i(b_2)<\varepsilon_i(b_1),\\ b_1\otimes e_ib_2, & \text{if } i=1,\cdots,n-1,\ \varphi_i(b_2)\geq\varepsilon_i(b_1),\\ e_0b_1\otimes b_2, & \text{if } \langle h_0,\mathrm{wt}(b_1)\rangle>0,\\ b_1\otimes e_0b_2, & \text{if } \langle h_0,\mathrm{wt}(b_1)\rangle=0,\end{cases}$$

$$f_i(b_1\otimes b_2)=\begin{cases} f_ib_1\otimes b_2, & \text{if } i=\overline{m-1},\cdots,\overline{1},\ \varphi_i(b_1)>\varepsilon_i(b_2),\\ b_1\otimes f_ib_2, & \text{if } i=\overline{m-1},\cdots,\overline{1},\ \varphi_i(b_1)\leq\varepsilon_i(b_2),\\ f_ib_1\otimes b_2, & \text{if } i=1,\cdots,n-1,\ \varphi_i(b_2)\leq\varepsilon_i(b_1),\\ b_1\otimes f_ib_2, & \text{if } i=1,\cdots,n-1,\ \varphi_i(b_2)>\varepsilon_i(b_1),\\ f_0b_1\otimes b_2, & \text{if } \langle h_0,\mathrm{wt}(b_1)\rangle>0,\\ b_1\otimes f_0b_2, & \text{if } \langle h_0,\mathrm{wt}(b_1)\rangle=0,\end{cases}$$

where we assume that $0\otimes b_2=b_1\otimes 0=0$.

Then, it is straightforward to check that $B_1\otimes B_2$ is a crystal graph for $\mathfrak{gl}_{m|n}$.

Definition 2.4. Let B_1 and B_2 be two crystal graphs for $\mathfrak{gl}_{m|n}$.

(1) The *direct sum* $B_1\oplus B_2$ is the disjoint union of B_1 and B_2.
(2) An *isomorphism* $\psi: B_1\to B_2$ of $\mathfrak{gl}_{m|n}$-crystals is an isomorphism of $I_{m|n}$-colored oriented graphs which preserves wt, ε_i, and φ_i ($i\in I_{m|n}$). We say that B_1 *is isomorphic to* B_2, and write $B_1\simeq B_2$.
(3) For $b_i\in B_i$ ($i=1,2$), let $C(b_i)$ denote the connected component of b_i as an $I_{m|n}$-colored oriented graph. We say that b_1 *is* $\mathfrak{gl}_{m|n}$*-equivalent to* b_2, if there is an isomorphism of crystal graphs $C(b_1)\to C(b_2)$ sending b_1 to b_2. We often write $b_1\simeq_{\mathfrak{gl}_{m|n}}b_2$ (or simply $b_1\simeq b_2$ if there is no confusion).

2.2. **Semistandard tableaux for $\mathfrak{gl}_{m|n}$.** $\mathbf{B}_{m|n}$ becomes a crystal graph for $\mathfrak{gl}_{m|n}$ whose associated $I_{m|n}$-colored oriented graph is given by

$$\overline{m}\ \stackrel{\overline{m-1}}{\longrightarrow}\ \overline{m-1}\ \stackrel{\overline{m-2}}{\longrightarrow}\cdots\stackrel{\overline{1}}{\longrightarrow}\ \overline{1}\ \stackrel{0}{\longrightarrow}\ 1\ \stackrel{1}{\longrightarrow}\cdots\stackrel{n-2}{\longrightarrow}n-1\stackrel{n-1}{\longrightarrow}n,$$

where $\mathrm{wt}(b)=\epsilon_b$, and $\varepsilon_i(b)$ (resp. $\varphi_i(b)$) is the number of i-colored arrows coming into b (resp. going out of b) for $i\in I_{m|n}$ and $b\in\mathbf{B}_{m|n}$. Note that $\mathbf{B}_{m|n}$ is the crystal graph associated to the natural representation $\mathbb{C}^{m|n}$.

Let $\mathcal{W}_{m|n}$ be the set of all finite words with the letters in $\mathbf{B}_{m|n}$. The empty word is denoted by $\emptyset$. Then $\mathcal{W}_{m|n}$ is a crystal graph for $\mathfrak{gl}_{m|n}$ since we may identify each non-empty word $w=w_1\cdots w_r$ with $w_1\otimes\cdots\otimes w_r\in\mathbf{B}_{m|n}^{\otimes r}$, where $\{\emptyset\}$ forms a trivial crystal graph, that is, $\mathrm{wt}(\emptyset)=0$, $e_i\emptyset=f_i\emptyset=0$, and $\varepsilon_i(\emptyset)=\varphi_i(\emptyset)=0$ for all $i\in I_{m|n}$.

Following the tensor product rule in Definition 2.3, we can describe the Kashiwara operators $e_i,f_i:\mathcal{W}_{m|n}\to\mathcal{W}_{m|n}\cup\{0\}$ ($i\in I_{m|n}$) in a more explicit way;

(1) Suppose that a non-empty word $w = w_1 \cdots w_r$ is given. To each letter w_k, we assign

$$\epsilon^{(i)}(w_k) = \begin{cases} +, & \text{if } (i=\overline{p},\, w_k=\overline{p+1}), \text{ or } (i=p,\, w_k=p), \text{ or } (i=0,\, w_k=\overline{1}), \\ -, & \text{if } (i=\overline{p},\, w_k=\overline{p}), \text{ or } (i=p,\, w_k=p+1), \text{ or } (i=0,\, w_k=1), \\ \,\cdot\,, & \text{otherwise,} \end{cases}$$

and let $\epsilon^{(i)}(w) = (\epsilon^{(i)}(w_1), \cdots, \epsilon^{(i)}(w_r))$.

(2) If $i = \overline{p}$ for $1 \le p \le m-1$, then we replace a pair $(\epsilon^{(i)}(w_s), \epsilon^{(i)}(w_{s'})) = (+,-)$ such that $s < s'$ and $\epsilon^{(i)}(w_t) = \cdot$ for $s < t < s'$ by $(\,\cdot\,,\,\cdot\,)$ in $\epsilon^{(i)}(w)$, and repeat this process as far as possible until we get a sequence with no $+$ placed to the left of $-$. If $i = p$ for $1 \le p \le n-1$, then we do the same work for $(-,+)$-pair in $\epsilon^{(i)}(w)$ until we get a sequence with no $-$ placed to the left of $+$. We call this sequence the *i-signature* of w. If $i = 0$, then we define *0-signature of w* to be $\epsilon^{(i)}(w_k)(\neq \cdot)$ such that $\epsilon^{(i)}(w_l) = \cdot$ for all $1 \le l < k$.

(3) If $i = \overline{p}$ (resp. $i = p$), then we call the right-most (resp. left-most) $-$ in the i-signature of w the *i-good $-$ sign*, and define $e_i w$ to be the word obtained by applying e_i to $\overline{p}$ (resp. $p+1$) corresponding to the i-good $-$ sign. If there is no i-good $-$ sign, then we define $e_i w = 0$.

(4) If $i = \overline{p}$ (resp. $i = p$), then we call the left-most (resp. right-most) $+$ in the i-signature of w the *i-good $+$ sign*, and define $f_i w$ to be the word obtained by applying f_i to $\overline{p+1}$ (resp. p) corresponding to the i-good $+$ sign. If there is no i-good $+$ sign, then we define $f_i w = 0$.

(5) If $i = 0$, then we define $e_0 w$ (resp. $f_0 w$) to be the word obtained by applying e_0 (resp. f_0) to the letter corresponding to the 0-signature of w. If the 0-signature of w is empty, then we define $e_0 w = 0$ (resp. $f_0 w = 0$).

Note that we have

$$\varepsilon_i(w) = \max\{\, k \mid e_i^k w \neq 0 \,\}, \quad \varphi_i(w) = \max\{\, k \mid f_i^k w \neq 0 \,\},$$

for $w \in \mathcal{W}_{m|n}$ and $i \in I_{m|n}$.

Example 2.5. Suppose that

$$w = 1\ \overline{1}\ \overline{1}\ \overline{1}\ \overline{2}\ 2\ 2\ \overline{2}\ 1.$$

Then

$$\begin{aligned} \epsilon^{(\overline{1})}(w) &= (\,\cdot\,, -, -, \ominus, \oplus, \,\cdot\,, \,\cdot\,, +, \,\cdot\,), \\ \epsilon^{(1)}(w) &= (\,\oplus, \,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot\,, \ominus, -, \,\cdot\,, +\,), \\ \epsilon^{(0)}(w) &= (\,\ominus, +, +, +, \,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot\,, -\,), \end{aligned}$$

where $\oplus, \ominus$ denote the i-good signs ($i \neq 0$), or the 0-signature. We have

$$\begin{aligned} e_{\overline{1}}(w) &= 1\ \overline{1}\ \overline{1}\ \overline{2}\ \overline{2}\ 2\ 2\ \overline{2}\ 1, & f_{\overline{1}}(w) &= 1\ \overline{1}\ \overline{1}\ \overline{1}\ \overline{1}\ 2\ 2\ \overline{2}\ 1, \\ e_1(w) &= 1\ \overline{1}\ \overline{1}\ \overline{1}\ \overline{2}\ 1\ 2\ \overline{2}\ 1, & f_1(w) &= 2\ \overline{1}\ \overline{1}\ \overline{1}\ \overline{2}\ 2\ 2\ \overline{2}\ 1, \\ e_0(w) &= \overline{1}\ \overline{1}\ \overline{1}\ \overline{1}\ \overline{2}\ 2\ 2\ \overline{2}\ 1, & f_0(w) &= 0. \end{aligned}$$

A *partition* is a non-increasing sequence of non-negative integers $\lambda = (\lambda_k)_{k \ge 1}$ such that all but a finite number of its terms are zero. Each λ_k is called a *part of* λ, and the

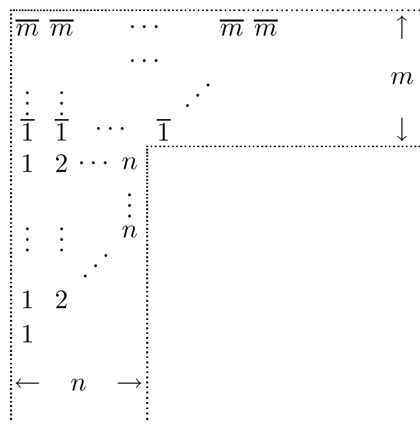


FIGURE 1. A highest weight tableau $H^{\lambda}_{m|n}$

number of non-zero parts is called the *length of* λ. We also write $\lambda = (1^{m_1}, 2^{m_2}, \cdots)$ where m_i is the number of occurrences of i in λ. Recall that a partition $\lambda = (\lambda_k)_{k\geq 1}$ is identified with a *Young diagram* which is a collection of nodes (or boxes) in left-justified rows with λ_k nodes in the k^{th} row.

A partition $\lambda = (\lambda_k)_{k\geq 1}$ is called an (m,n)-*hook partition* if $\lambda_{m+1} \leq n$. We denote by $\mathcal{P}_{m|n}$ the set of all (m,n)-hook partitions. A tableau T obtained by filling a Young diagram λ with the entries in $\mathbf{B}_{m|n}$ is called (m,n)-*hook semistandard* if

(1) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom),
(2) the entries in $\mathbf{B}^{+}_{m|n}$ (resp. $\mathbf{B}^{-}_{m|n}$) are strictly increasing in each column (resp. row)

(see [2]). We say that λ is the *shape of* T. It is easy to see that a partition λ can be made into an (m,n)-hook semistandard tableau if and only if $\lambda \in \mathcal{P}_{m|n}$.

For $\lambda \in \mathcal{P}_{m|n}$, let $\mathbf{B}_{m|n}(\lambda)$ be the set of all (m,n)-hook semistandard tableaux of shape λ. We may view $\mathbf{B}_{m|n}(\lambda)$ as a subset of $\mathcal{W}_{m|n}$ by column reading (or far eastern reading). That is, we read the entries of a tableau column by column from right to left, and in each column we read the entries from top to bottom.

For $T \in \mathbf{B}_{m|n}(\lambda)$, the weight of T is given by $\operatorname{wt} T = \sum_{b\in\mathbf{B}_{m|n}} \mu_b \epsilon_b \in P_{m|n}$, where μ_b is number of occurrences of b in T. Indeed, $\mathbf{B}_{m|n}(\lambda)$ together with 0 is stable under e_i, f_i $(i \in I_{m|n})$, and $\mathbf{B}_{m|n}(\lambda)$ is a subcrystal of $\mathcal{W}_{m|n}$.

Theorem 2.6 ([1]). *For $\lambda \in \mathcal{P}_{m|n}$, $\mathbf{B}_{m|n}(\lambda)$ is a crystal graph for $\mathfrak{gl}_{m|n}$. Moreover, $\mathbf{B}_{m|n}(\lambda)$ is a connected $I_{m|n}$-colored oriented graph with a unique highest weight element $H^{\lambda}_{m|n}$.* □

Remark 2.7. Note that $\operatorname{wt}(H^{\lambda}_{m|n}) \geq \operatorname{wt}(T)$ for all $T \in \mathbf{B}_{m|n}(\lambda)$ ([1]) (see Figure 1), and hence $e_i H^{\lambda}_{m|n} = 0$ for all $i \in I_{m|n}$. But, unlike the crystal graphs associated to integrable highest weight representations of a symmetrizable Kac-Moody algebra, there might exist $T \in \mathbf{B}_{m|n}(\lambda)$ such that $T \neq H^{\lambda}_{m|n}$ and $e_i T = 0$ for all $i \in I_{m|n}$.

Such a tableau T was called a *fake highest weight vector*, and $H^{\lambda}_{m|n}$ a *genuine highest weight vector* in [1].

To characterize a connected component in $\mathcal{W}_{m|n}$, we need the algorithm of *Schensted's column bumping* for (m,n)-hook semistandard tableaux ([2, 16]): for $\lambda \in \mathcal{P}_{m|n}$ and $T \in \mathbf{B}_{m|n}(\lambda)$, we define $T \leftarrow b$ ($b \in \mathbf{B}_{m|n}$) to be the tableau obtained from T by applying the following procedure;

(1) If $b \in \mathbf{B}^{+}_{m|n}$, let b' be the smallest entry in the first (or the left-most) column which is greater than or equal to b. If $b \in \mathbf{B}^{-}_{m|n}$, let b' be the smallest entry in the first column which is greater than b. If there are more than one b', choose the one in the highest position.
(2) Replace b' by b (b' is *bumped out* of the first column). If there is no such b', put b at the bottom of the first column and stop the procedure.
(3) Repeat (1) and (2) on the next column with b'.

Note that $(T \leftarrow b) \in \mathbf{B}_{m|n}(\mu)$ for some $\mu \in \mathcal{P}_{m|n}$, where μ is given by adding a node at λ. Now, for a given word $w = w_1 \cdots w_r \in \mathcal{W}$, we define

$$P(w) = (\cdots((w_1 \leftarrow w_2\,) \leftarrow w_3\,)\cdots) \leftarrow w_r\ . \tag{2.1}$$

Lemma 2.8 ([1]). *For $w \in \mathcal{W}_{m|n}$, we have $w \simeq P(w)$. Hence, any connected component in $\mathcal{W}_{m|n}$ is isomorphic to $\mathbf{B}_{m|n}(\lambda)$ for some $\lambda \in \mathcal{P}_{m|n}$.* □

We will also use the following lemma in the next section.

Lemma 2.9 ([11]). *Let T and T' be two (m,n)-hook semistandard tableaux. If $T \simeq T'$, then $T = T'$.* □

Let $x = \{\, x_b \,|\, b \in \mathbf{B}_{m|n} \,\}$ be the set of variables indexed by $\mathbf{B}_{m|n}$. For $\mu = \sum_{b\in\mathbf{B}_{m|n}} \mu_b \epsilon_b \in P_{m|n}$, we set $x^{\mu} = \prod_{b\in\mathbf{B}_{m|n}} x_b^{\mu_b}$. For $\lambda \in \mathcal{P}_{m|n}$, we define a *hook Schur function* corresponding to λ by

$$hs_{\lambda}(x) = \sum_{T\in\mathbf{B}_{m|n}(\lambda)} x^{\mathrm{wt}T},$$

which is the character of $\mathbf{B}_{m|n}(\lambda)$ (see [2, 16]).

3. Bicrystal graphs and Cauchy decomposition

We consider a bicrystal graph structure on the set of certain matrices of non-negative integers, which parameterizes the monomial basis of $S(\mathbb{C}^{m|n} \otimes \mathbb{C}^{u|v})$, and then derive an explicit decomposition by finding all the highest weight elements.

3.1. **Crystal graphs of biwords.** Suppose that m,n,u,v are non-negative integers such that $m+n, u+v > 0$.

Let

$$\begin{aligned}\Omega_{m|n,u|v} = \{\, (\mathbf{i},\mathbf{j}) \in \mathcal{W}_{m|n} \times \mathcal{W}_{u|v} \,|\, & \\ & (1)\ \mathbf{i} = i_1,\cdots,i_r \text{ and } \mathbf{j} = j_1,\cdots,j_r \text{ for some } r \geq 0, \\ & (2)\ (i_1,j_1) \leq \cdots \leq (i_r,j_r), \\ & (3)\ |i_k| \neq |j_k| \text{ implies } (i_k,j_k) \neq (i_{k\pm1},j_{k\pm1})\},\end{aligned} \tag{3.1}$$

where for (i,j) and $(k,l) \in \mathbf{B}_{m|n} \times \mathbf{B}_{u|v}$, the *super lexicographic ordering* is given by

$$(3.2) \qquad (i,j) < (k,l) \quad \Leftrightarrow \quad \begin{cases} (j < l) & \text{or,} \\ (j = l \in \mathbf{B}^{+}_{u|v}, \text{ and } i > k) & \text{or,} \\ (j = l \in \mathbf{B}^{-}_{u|v}, \text{ and } i < k) & . \end{cases}$$

Next, let $\Omega^{*}_{m|n,u|v}$ be the set of pairs $(\mathbf{k},\mathbf{l}) \in \mathcal{W}_{m|n} \times \mathcal{W}_{u|v}$ such that $(\mathbf{l},\mathbf{k}) \in \Omega_{u|v,m|n}$. For simplicity, we write $\Omega = \Omega_{m|n,u|v}$ and $\Omega^{*} = \Omega^{*}_{m|n,u|v}$.

Now, for $i \in I_{m|n}$ and $(\mathbf{i},\mathbf{j}) \in \Omega$, we define

$$e_i(\mathbf{i},\mathbf{j}) = (e_i\mathbf{i},\mathbf{j}), \qquad f_i(\mathbf{i},\mathbf{j}) = (f_i\mathbf{i},\mathbf{j}),$$

where we assume that $x_i(\mathbf{i},\mathbf{j}) = 0$ if $x_i\mathbf{i} = 0$ $(x = e,f)$. We set $\mathrm{wt}(\mathbf{i},\mathbf{j}) = \mathrm{wt}(\mathbf{i})$, $\varepsilon_i(\mathbf{i},\mathbf{j}) = \varepsilon_i(\mathbf{i})$ and $\varphi_i(\mathbf{i},\mathbf{j}) = \varphi_i(\mathbf{i})$ $(i \in I_{m|n})$.

Similarly, for $i \in I_{u|v}$ and $(\mathbf{k},\mathbf{l}) \in \Omega^{*}$, we define

$$e^{*}_j(\mathbf{k},\mathbf{l}) = (\mathbf{k},e_j\mathbf{l}), \qquad f^{*}_j(\mathbf{k},\mathbf{l}) = (\mathbf{k},f_j\mathbf{l}),$$

and set $\mathrm{wt}^{*}(\mathbf{k},\mathbf{l}) = \mathrm{wt}(\mathbf{l})$, $\varepsilon^{*}_j(\mathbf{k},\mathbf{l}) = \varepsilon_j(\mathbf{l})$ and $\varphi^{*}_j(\mathbf{k},\mathbf{l}) = \varphi_j(\mathbf{l})$ $(j \in I_{u|v})$.

Lemma 3.1. *Under the above hypothesis,*

(1) *the set Ω together with $\mathrm{wt}, e_i, f_i, \varepsilon_i, \varphi_i$ $(i \in I_{m|n})$ is a crystal graph for $\mathfrak{gl}_{m|n}$,*
(2) *the set Ω^{*} together with $\mathrm{wt}^{*}, e^{*}_j, f^{*}_j, \varepsilon^{*}_j, \varphi^{*}_j$ $(j \in I_{u|v})$ is a crystal graph for $\mathfrak{gl}_{u|v}$.*

Proof. We will prove only (1) since the proof of (2) is the same. Suppose that $(\mathbf{i},\mathbf{j}) \in \Omega \setminus \{(\emptyset,\emptyset)\}$ is given where $\mathbf{i} = i_1 \cdots i_r$ and $\mathbf{j} = j_1 \cdots j_r$ for some $r \geq 1$. We write

$$\mathbf{i} = \mathbf{i}_{\overline{u}}\mathbf{i}_{\overline{u-1}} \cdots \mathbf{i}_{v-1}\mathbf{i}_v,$$

where $\mathbf{i}_b = i_{t_1} \cdots i_{t_b}$ $(b \in \mathbf{B}_{u|v})$ is a subword of $\mathbf{i}$ such that $j_{t_1} = \cdots = j_{t_b} = b$.

Then, $\mathbf{i}_b$ $(b \in \mathbf{B}_{u|v})$ is $\mathfrak{gl}_{m|n}$-equivalent to an (m,n)-hook semistandard tableau T_b of a single row or a single column as follows;

$$T_b = \begin{cases} i_{t_1} \cdots i_{t_b} \quad \in \mathbf{B}_{m|n}((t_b)), & \text{if } b \in \mathbf{B}^{+}_{u|v}, \\ \begin{matrix} i_1 \\ \vdots \\ i_{t_b} \end{matrix} \quad \in \mathbf{B}_{m|n}((1^{t_b})), & \text{if } b \in \mathbf{B}^{-}_{u|v}. \end{cases}$$

So, $\mathbf{i}$ is $\mathfrak{gl}_{m|n}$-equivalent to

$$T_{\overline{u}} \otimes T_{\overline{u-1}} \otimes \cdots \otimes T_{v-1} \otimes T_v,$$

where $T_b = \emptyset$ if $\mathbf{i}_b = \emptyset$. Therefore, if $x_i(\mathbf{i},\mathbf{j}) \neq 0$ $(x = e,f)$ for $i \in I_{m|n}$, then $(x_i\mathbf{i},\mathbf{j}) \in \Omega$. It follows that Ω is a crystal graph for $\mathfrak{gl}_{m|n}$. $\square$

3.2. **Bicrystal graphs.** Consider the following set of matrices of non-negative integers;

$$(3.3) \qquad \begin{aligned} \mathbf{M}_{m|n,u|v} = \{\, A =& (a_{bb'})_{b \in \mathbf{B}_{m|n}, b' \in \mathbf{B}_{u|v}} \mid \\ & (1)\ a_{bb'} \in \mathbb{Z}_{\geq 0},\ \ (2)\ a_{bb'} \leq 1 \text{ if } |b| \neq |b'| \,\}. \end{aligned}$$

For simplicity, we write $\mathbf{M} = \mathbf{M}_{m|n,u|v}$.

For $(\mathbf{i},\mathbf{j})\in\Omega$, define $A(\mathbf{i},\mathbf{j})=(a_{bb'})$ to be the matrix in $\mathbf{M}$, where $a_{bb'}$ is the number of k's such that $(i_k,j_k)=(b,b')$ for $b\in\mathbf{B}_{m|n}$ and $b'\in\mathbf{B}_{u|v}$. Then, it follows that the map $(\mathbf{i},\mathbf{j})\mapsto A(\mathbf{i},\mathbf{j})$ gives a bijection between Ω and $\mathbf{M}$, where the pair of empty words $(\emptyset,\emptyset)$ corresponds to zero matrix. Similarly, we have a bijection $(\mathbf{k},\mathbf{l})\mapsto A(\mathbf{k},\mathbf{l})$ from Ω^* to $\mathbf{M}$.

With these identifications, $\mathbf{M}$ becomes a crystal graph for both $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_{u|v}$ by Lemma 3.1.

Example 3.2. Suppose that $m|n=u|v=2|2$ and

$$A=\left(\begin{array}{cc|cc} 1 & 0 & 1 & 1\\ 0 & 2 & 0 & 0\\ \hline 1 & 0 & 0 & 1\\ 0 & 0 & 2 & 0\end{array}\right)\in\mathbf{M}.$$

Then $A=A(\mathbf{i},\mathbf{j})=A(\mathbf{k},\mathbf{l})$ for $(\mathbf{i},\mathbf{j})\in\Omega$ and $(\mathbf{k},\mathbf{l})\in\Omega^*$, where

$$\begin{aligned}\mathbf{i}&=1\ \overline{2}\ \overline{1}\ \overline{1}\ \overline{2}\ 2\ 2\ \overline{2}\ 1, &\quad \mathbf{k}&=\overline{2}\ \overline{2}\ \overline{2}\ \overline{1}\ \overline{1}\ 1\ 1\ 2\ 2,\\ \mathbf{j}&=\overline{2}\ \overline{2}\ \overline{1}\ \overline{1}\ 1\ 1\ 1\ 2\ 2, &\quad \mathbf{l}&=2\ 1\ \overline{2}\ \overline{1}\ \overline{1}\ \overline{2}\ 2\ 1\ 1.\end{aligned}$$

Definition 3.3. Let B be a crystal graph for both $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_{u|v}$. We denote by e_j^* and f_j^* $(j\in I_{u|v})$ the Kashiwara operators for $\mathfrak{gl}_{u|v}$. We call B a *crystal graph for* $\mathfrak{gl}_{m|n}\oplus\mathfrak{gl}_{u|v}$*, or* $(\mathfrak{gl}_{m|n},\mathfrak{gl}_{u|v})$*-bicrystal* if e_i,f_i commute with e_j^*,f_j^* $(i\in I_{m|n}, j\in I_{u|v})$, where we understand the Kashiwara operators as the associated maps from $B\cup\{0\}$ to itself (that is, $x_i0=x_j^*0=0$, for $x=e,f$).

For example, $B=\mathbf{B}_{m|n}(\lambda)\times\mathbf{B}_{u|v}(\mu)$ $(\lambda\in\mathcal{P}_{m|n},\ \mu\in\mathcal{P}_{u|v})$ is a $(\mathfrak{gl}_{m|n},\mathfrak{gl}_{u|v})$-bicrystal where $x_i(b_1,b_2)=(x_ib_1,b_2)$, $x_j^*(b_1,b_2)=(b_1,x_j^*b_2)$ for $(b_1,b_2)\in B$ and $x=e,f$.

Lemma 3.4. $\mathbf{M}$ *is a* $(\mathfrak{gl}_{m|n},\mathfrak{gl}_{u|v})$*-bicrystal.*

Proof. The proof is a straightforward verification. So, let us prove the following case;

$$e_t^*e_sA=e_se_t^*A \tag{3.4}$$

for $A\in\mathbf{M}$, $1\le s\le n-1$ and $1\le t\le v-1$. The other cases can be checked in a similar manner.

To show this, we may assume that $A=(a_{bb'})$ such that $a_{bb'}=0$ unless $b\in\{s,s+1\}$ or $b'\in\{t,t+1\}$. Put $A'=(a_{bb'})_{b=s,s+1}$ and $A''=(a_{bb'})_{b'=t,t+1}$, which are the submatrices of A. If neither A' nor A'' contains both an s-good $-$ sign and a t-good $-$ sign of A, then (3.4) follows from the fact that the s-signature of e_t^*A is the same as that of A, and the t-signature of A is the same as that of e_sA. So we may assume that A is either A' or A'', and prove the case only when $A=A'$, that is,

$$A=\begin{pmatrix} x_{\overline{u}}^+ & \cdots & x_{\overline{1}}^+ & x_1^+ & \cdots & x_v^+\\ x_{\overline{u}}^- & \cdots & x_{\overline{1}}^- & x_1^- & \cdots & x_v^-\end{pmatrix},$$

where $x_b^+ = a_{sb}$ and $x_b^- = a_{s+1b}$ ($b \in \mathbf{B}_{u|v}$). Note that $A = A(\mathbf{i},\mathbf{j}) = A(\mathbf{k},\mathbf{l})$ for unique $(\mathbf{i},\mathbf{j}) \in \Omega$ and $(\mathbf{k},\mathbf{l}) \in \Omega^*$. Then $\mathbf{i}$ and $\epsilon^{(s)}(\mathbf{i})$ are of the form:

$$\mathbf{i} = (s+1)^{x_{\overline{u}}^-}\, s^{x_{\overline{u}}^+} \cdots (s+1)^{x_{\overline{1}}^-}\, s^{x_{\overline{1}}^+}\, s^{x_1^+}\, (s+1)^{x_1^-} \cdots s^{x_v^+}\, (s+1)^{x_v^-},$$

$$\epsilon^{(s)}(\mathbf{i}) = (-^{x_{\overline{u}}^-}, +^{x_{\overline{u}}^+}, \cdots, -^{x_{\overline{1}}^-}, +^{x_{\overline{1}}^+}, +^{x_1^+}, -^{x_1^-}, \cdots, +^{x_v^+}, -^{x_v^-}),$$

where the multiplicities of letters and signs are given as exponents. On the other hand, the t-signature of A or $\mathbf{l}$ is completely determined by its subword $\mathbf{l}'$:

$$\mathbf{l}' = t^{x_t^+}\,(t+1)^{x_{t+1}^+}\,t^{x_t^-}\,(t+1)^{x_{t+1}^-},$$

$$\epsilon^{(t)}(\mathbf{l}') = (+^{x_t^+}, -^{x_{t+1}^+}, +^{x_t^-}, -^{x_{t+1}^-}).$$

For convenience, we set

$$\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x_t^+ & x_{t+1}^+ \\ x_t^- & x_{t+1}^- \end{pmatrix}.$$

CASE 1. Suppose that $e_t^* A = 0$ (equivalently, $e_t^* \tilde{A} = 0$). This implies that $d = 0$ and $b \le c$ (note that in this case, we cancel out $(-,+)$ pairs to obtain a t-signature). If an s-good $-$ sign occurs in $\tilde{A}$, then we have $b < c$ and

$$e_s A = \begin{pmatrix} \cdots & a+1 & b & \cdots \\ \cdots & c-1 & 0 & \cdots \end{pmatrix}.$$

Since $b \le c-1$, we get $e_t^* e_s A = 0$, which implies (3.4). If an s-good $-$ sign of A does not occur in $\tilde{A}$, then $\tilde{A}$ does not change when we apply e_s to A and $e_t^* e_s A = 0 = e_s e_t^* A$, which also implies (3.4).

CASE 2. Suppose that $e_t^* A \ne 0$ and b is changed by e_t^*. This implies that $b > 0$ and $b > c$. If $e_t^* A = A(\mathbf{i}',\mathbf{j}')$ for $(\mathbf{i}',\mathbf{j}') \in \Omega$, then

$$\epsilon^{(s)}(\mathbf{i}) = (\cdots, +^a, -^c, +^b, -^d, \cdots),$$

$$\epsilon^{(s)}(\mathbf{i}') = (\cdots, +^{a+1}, -^c, +^{b-1}, -^d, \cdots).$$

Note that the subsequences $(+^{a+1}, -^c, +^{b-1}, -^d)$ and $(+^a, -^c, +^b, -^d)$ reduce to the same sequence $(+^{a+b-c}, -^d)$ $(a+b-c > 0)$, and there is no s-good $-$ sign in c. If no s-good $-$ sign of A occurs in $\tilde{A}$ (or, in d), then it is easy to see that (3.4) holds. If there is an s-good $-$ sign in $\tilde{A}$, then

$$e_s e_t^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e_s \begin{pmatrix} a+1 & b-1 \\ c & d \end{pmatrix} = \begin{pmatrix} a+1 & b \\ c & d-1 \end{pmatrix},$$

$$e_t^* e_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e_t^* \begin{pmatrix} a & b+1 \\ c & d-1 \end{pmatrix} = \begin{pmatrix} a+1 & b \\ c & d-1 \end{pmatrix}.$$

Hence, we have (3.4).

CASE 3. Suppose that $e_t^* A \ne 0$ and d is changed by e_t^*. This implies that $d > 0$ and $b \le c$. If $e_t^* A = A(\mathbf{i}',\mathbf{j}')$ for $(\mathbf{i}',\mathbf{j}') \in \Omega$, then

$$\epsilon^{(s)}(\mathbf{i}) = (\cdots, +^a, -^c, +^b, -^d, \cdots),$$

$$\epsilon^{(s)}(\mathbf{i}') = (\cdots, +^a, -^{c+1}, +^b, -^{d-1}, \cdots).$$

Note that the subsequences $(+^a, -^{c+1}, +^b, -^{d-1})$ and $(+^a, -^c, +^b, -^d)$ reduce to the same sequence $(+^a, -^{c-b+d})$ $(c-b+d>0)$. If no s-good $-$ sign of A occurs in $\tilde{A}$ (or, in d), then it is easy to see that $e_t^* e_s A = e_s e_t^* A$. Assume that an s-good $-$ sign of A occurs in $\tilde{A}$. If $b=c$, then we have

$$e_s e_t^* \begin{pmatrix} a & b \\ b & d \end{pmatrix} = e_s \begin{pmatrix} a & b \\ b+1 & d-1 \end{pmatrix} = \begin{pmatrix} a+1 & b \\ b & d-1 \end{pmatrix},$$
$$e_t^* e_s \begin{pmatrix} a & b \\ b & d \end{pmatrix} = e_t^* \begin{pmatrix} a & b+1 \\ b & d-1 \end{pmatrix} = \begin{pmatrix} a+1 & b \\ b & d-1 \end{pmatrix}.$$

If $b<c$, then we have

$$e_s e_t^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e_s \begin{pmatrix} a & b \\ c+1 & d-1 \end{pmatrix} = \begin{pmatrix} a+1 & b \\ c & d-1 \end{pmatrix},$$
$$e_t^* e_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e_t^* \begin{pmatrix} a+1 & b \\ c-1 & d \end{pmatrix} = \begin{pmatrix} a+1 & b \\ c & d-1 \end{pmatrix}.$$

Hence, we have (3.4). □

Lemma 3.5.

(1) *Let C be a connected component in $\mathbf{M}$ as a $\mathfrak{gl}_{m|n}$-crystal. If $x_j^* C \neq \{0\}$ for some $j \in I_{u|v}$ and $x = e, f$, then $x_j^* : C \to x_j^* C$ is an isomorphism of $\mathfrak{gl}_{m|n}$-crystals.*

(2) *Let C^* be a connected component in $\mathbf{M}$ as a $\mathfrak{gl}_{u|v}$-crystal. If $x_i C^* \neq \{0\}$ for some $i \in I_{m|n}$ and $x = e, f$, then $x_i : C^* \to x_i C^*$ is an isomorphism of $\mathfrak{gl}_{u|v}$-crystals.*

Proof. (1) Choose $b \in C$ such that $x_j^* b \neq 0$. For any $b' \in C$, we have

$$b' = x_{i_1} \cdots x_{i_t} b,$$

for some Kashiwara operators x_{i_k} $(i_k \in I_{m|n}, 1 \leq k \leq t)$. We first claim that $x_j^* b' \neq 0$.

We will use induction on t. Suppose that $t=1$. Let y_{i_1} be the Kashiwara operator given by $y_{i_1} = f_{i_1}$ (resp. $y_{i_1} = e_{i_1}$) if $x_{i_1} = e_{i_1}$ (resp. $x_{i_1} = f_{i_1}$). Then $b = y_{i_1} b'$, and

$$0 \neq x_j^* b = x_j^* y_{i_1} b' = y_{i_1} x_j^* b'$$

by Lemma 3.4, which implies that $x_j^* b' \neq 0$. Suppose that $t>1$. Since $x_j^* x_{i_t} b \neq 0$ and $b' = x_{i_1} \cdots x_{i_{t-1}} (x_{i_t} b)$, it follows that $x_j^* b' \neq 0$ by induction hypothesis. This completes the induction.

Hence, the composite of the following two maps is the identity map on C;

$$C \xrightarrow{x_j^*} x_j^* C \xrightarrow{y_j^*} C,$$

where $y_j^* = e_j^*$ (resp. f_j^*) if $x_j^* = f_j^*$ (resp. e_j^*). This implies that x_j^* is a bijection which commutes with e_i, f_i $(i \in I_{m|n})$ by Lemma 3.4, and that $x_j^* C$ is isomorphic to C as a $\mathfrak{gl}_{m|n}$-crystal. The proof of (2) is similar. □

Given $A \in \mathbf{M}$, suppose that $A = A(\mathbf{i}, \mathbf{j}) = A(\mathbf{k}, \mathbf{l})$ for unique $(\mathbf{i}, \mathbf{j}) \in \Omega$ and $(\mathbf{k}, \mathbf{l}) \in \Omega^*$. We define

$$\pi(A) = (P_1(A), P_2(A)) = (P(\mathbf{i}), P(\mathbf{l})), \tag{3.5}$$

(see (2.1)). Then $\pi(A) \in \mathbf{B}_{m|n}(\lambda) \times \mathbf{B}_{u|v}(\mu)$ for some $(\lambda, \mu) \in \mathcal{P}_{m|n} \times \mathcal{P}_{u|v}$. Note that $A \simeq_{\mathfrak{gl}_{m|n}} P_1(A)$, and $A \simeq_{\mathfrak{gl}_{u|v}} P_2(A)$.

Lemma 3.6. *Suppose that $A \in \mathbf{M}$ is given.*

(1) *if $x_j^* A \neq 0$ for some $j \in I_{u|v}$ and $x = e, f$, then $P_1(x_j^* A) = P_1(A)$.*
(2) *if $x_i A \neq 0$ for some $i \in I_{m|n}$ and $x = e, f$, then $P_2(x_i A) = P_2(A)$.*

Proof. It suffices to prove (1). By Lemma 3.5, A is $\mathfrak{gl}_{m|n}$-equivalent to $x_j^* A$, and hence the tableaux $P_1(A)$ and $P_1(x_j^* A)$ are $\mathfrak{gl}_{m|n}$-equivalent. By Lemma 2.9, we have $P_1(x_j^* A) = P_1(A)$. □

Let B be a $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$-bicrystal. If we set $\mathcal{I} = I_{m|n} \sqcup I_{u|v}^*$ where $I_{u|v}^* = \{j^* | j \in I_{u|v}\}$, then B is an $\mathcal{I}$-colored oriented graph with respect to x_i, x_j^* for $i \in I_{m|n}, j \in I_{u|v}$, and $x = e, f$. Now, we can characterize a connected component in $\mathbf{M}$ as a $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$-bicrystal.

Proposition 3.7. *For each connected component $\mathbf{C}$ in $\mathbf{M}$, π gives the following isomorphism of $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$-bicrystals;*

$$\pi : \mathbf{C} \longrightarrow \mathbf{B}_{m|n}(\lambda) \times \mathbf{B}_{u|v}(\mu),$$

for some $\lambda \in \mathcal{P}_{m|n}$ and $\mu \in \mathcal{P}_{u|v}$.

Proof. By Lemma 3.6, we have

$$\pi(x_i A) = (x_i P_1(A), P_2(A)),$$

for $A \in \mathbf{M}$, $i \in I_{m|n}$ and $x = e, f$ (we assume that $\pi(0) = 0$ and $(0, P_2(A)) = 0$). Similarly, we have $\pi(x_j^* A) = (P_1(A), x_j P_2(A))$ for $j \in I_{u|v}$.

Let $\mathbf{C}$ be a connected component in $\mathbf{M}$ as an $\mathcal{I}$-colored oriented graph. Choose an arbitrary $A \in \mathbf{C}$. Suppose that $\pi(A) \in \mathbf{B}_{m|n}(\lambda) \times \mathbf{B}_{u|v}(\mu)$ for some $\lambda \in \mathcal{P}_{m|n}$ and $\mu \in \mathcal{P}_{u|v}$. Then we have

$$\pi : \mathbf{C} \longrightarrow \mathbf{B}_{m|n}(\lambda) \times \mathbf{B}_{u|v}(\mu),$$

where π commutes with e_i, f_i and e_j^*, f_j^* $(i \in I_{m|n}, j \in I_{u|v})$. It is clear that π is onto.

Now suppose that $\pi(A) = \pi(A')$ for some $A, A' \in \mathbf{C}$. Since $\mathbf{C}$ is connected,

$$x_{j_1}^* \cdots x_{j_q}^* x_{i_1} \cdots x_{i_p} A = A',$$

for some Kashiwara operators x_{i_k} $(i_k \in I_{m|n},\ 1 \le k \le p)$, $x_{j_l}^*$ $(j_l \in I_{u|v},\ 1 \le l \le q)$. Put $A'' = x_{i_1} \cdots x_{i_p} A$. Then A and A'' belong to the same connected component as a $\mathfrak{gl}_{m|n}$-crystal, say C_1. On the other hand, by Lemma 3.6, we have

$$P_1(A'') = P_1(A') = P_1(A).$$

Since the map $P_1 : C_1 \to \mathbf{B}_{m|n}(\lambda)$ is an isomorphism of $\mathfrak{gl}_{m|n}$-crystals, it follows that $A'' = A$.

Next, A'' and A' belong to the same connected component as a $\mathfrak{gl}_{u|v}$-crystal, say C_2. By Lemma 3.6, we have

$$P_2(A'') = P_2(A) = P_2(A').$$

Since the map $P_2 : C_2 \to \mathbf{B}_{u|v}(\mu)$ is an isomorphism, we have $A'' = A'$, and hence $A = A'$. So, π is one-to-one.

Therefore, π is an isomorphism of $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$-bicrystals. □

Example 3.8. Let A be as in Example 3.2. Then

$$\pi(A) = \left(\begin{array}{lllll} \overline{2} & \overline{2} & \overline{2} & \overline{1} & 1 \\ \overline{1} & 2 & & & \\ 1 & & & & \\ 2 & & & & \end{array} , \begin{array}{lllll} \overline{2} & \overline{2} & \overline{1} & 1 & 2 \\ \overline{1} & 2 & & & \\ 1 & & & & \\ 2 & & & & \end{array} \right),$$

and the connected component of A is isomorphic to $\mathbf{B}_{2|2}(5,2,1,1) \times \mathbf{B}_{2|2}(5,2,1,1)$.

3.3. **Decomposition of M.** Now, we will describe an explicit decomposition of $\mathbf{M}$. Set

$$\mathbf{M}_{\text{h.w.}} = \{\, A \in \mathbf{M} \,|\, \pi(A) = (H^{\lambda}_{m|n}, H^{\mu}_{u|v}) \text{ for some } (\lambda,\mu) \in \mathcal{P}_{m|n} \times \mathcal{P}_{u|v} \,\}, \tag{3.6}$$

which is the set of all the highest weight elements in $\mathbf{M}$. We have seen in the proof of Proposition 3.7 that π induces an isomorphism of $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$-bicrystals between the connected components of $A \in \mathbf{M}$ and $\pi(A)$. Hence,

$$\mathbf{M} = \bigoplus_{A \in \mathbf{M}_{\text{h.w.}}} \mathbf{C}(A),$$

where $\mathbf{C}(A)$ is the connected component of A in $\mathbf{M}$, which is isomorphic to $\mathbf{B}_{m|n}(\lambda) \times \mathbf{B}_{u|v}(\mu)$ for some $\lambda \in \mathcal{P}_{m|n}$ and $\mu \in \mathcal{P}_{u|v}$.

Suppose that $\lambda = (\lambda_1, \cdots, \lambda_r) \in \mathcal{P}_{m|n} \cap \mathcal{P}_{u|v}$ is given. Let $\nu = (\nu_1, \nu_2, \cdots, \nu_\ell)$ be the sequence of non-negative integers ($\ell = \lambda_{m+1}$) determined by

$$(\nu_1 + \cdots + \nu_\ell, \nu_2 + \cdots + \nu_\ell, \cdots, \nu_\ell) = (\lambda_{m+1}, \cdots, \lambda_r)', \tag{3.7}$$

where $(\lambda_{m+1}, \cdots, \lambda_r)'$ is the conjugate of the partition $(\lambda_{m+1}, \cdots, \lambda_r)$ (cf.[15]).

Assume that $m \geq u$. Let us define $A_\lambda = (a_{bb'}) \in \mathbf{M}$ by

(1) for $0 \leq k < m$, $0 \leq l < u$,

$$a_{\overline{m-k}\,\overline{u-l}} = \begin{cases} \lambda_{k+1}, & \text{if } 0 \leq k = l < u, \\ 0, & \text{otherwise.} \end{cases}$$

(2) for $0 \leq k < m$, $1 \leq t \leq v$,

$$a_{\overline{m-k}\,t} = \begin{cases} 1, & \text{if } u \leq k < m \text{ and } \lambda_{k+1} \geq t, \\ 0, & \text{otherwise.} \end{cases}$$

(3) for $1 \leq s \leq n$, $0 \leq l < u$, $\quad a_{s\,\overline{u-l}} = 0$.

(4) for $1 \leq s \leq n$, $1 \leq t \leq v$,

$$a_{s\,t} = \begin{cases} \nu_{s+t-1}, & \text{if } 2 \leq s+t \leq \ell+1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that A_λ is of the following form;

$$A_\lambda = \left(\begin{array}{ccccc|ccccc} \lambda_1 &&&&&&&&& \\ & \lambda_2 &&&&&&&& \\ && \ddots &&&&&&& \\ &&& \lambda_{u-1} &&&&&& \\ &&&& \lambda_u &&&&& \\ &&&&& 1 & \cdots & \cdots & \cdots & 1 \\ &&&&& \vdots &&&& \\ &&&&& 1 & \cdots & \cdots & 1 & \\ \hline &&&&& \nu_1 & \nu_2 & \cdots & \nu_\ell & \\ &&&&& \nu_2 & \nu_3 & \cdots && \\ &&&&& \vdots &&&& \\ &&&&& \nu_\ell &&&& \end{array}\right). \tag{3.8}$$

If $m \leq u$, then we define A_λ to be the transpose of (3.8), where m and u are exchanged. Note that A_λ is uniquely determined by λ. By Schensted's column bumping algorithm, it is not difficult to see that

Lemma 3.9. *For $\lambda \in \mathcal{P}_{m|n} \cap \mathcal{P}_{u|v}$, we have*

$$\pi(A_\lambda) = (H^\lambda_{m|n}, H^\lambda_{u|v}) \in \mathbf{B}_{m|n}(\lambda) \times \mathbf{B}_{u|v}(\lambda).$$

$\square$

Example 3.10. Suppose that $m|n = 4|3$ and $u|v = 3|3$. Let $\lambda = (7, 5, 5, 3, 3, 2, 2, 1)$. Then

$$A_\lambda = \left(\begin{array}{ccc|ccc} 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right),$$

and

$$\pi(A_\lambda) = \left(\begin{array}{ccccccc} \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} \\ \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} && \\ \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} && \\ \overline{1} & \overline{1} & \overline{1} &&&& \\ 1 & 2 & 3 &&&& \\ 1 & 2 &&&&& \\ 1 & 2 &&&&& \\ 1 &&&&&& \end{array}, \begin{array}{ccccccc} \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} \\ \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} && \\ \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} && \\ 1 & 2 & 3 &&&& \\ 1 & 2 & 3 &&&& \\ 1 & 2 &&&&& \\ 1 & 2 &&&&& \\ 1 &&&&&& \end{array}\right).$$

Theorem 3.11. *We have*

$$\mathbf{M}_{\text{h.w.}} = \{\, A_\lambda \mid \lambda \in \mathcal{P}_{m|n} \cap \mathcal{P}_{u|v} \,\},$$

and hence the following isomorphism of $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$-bicrystals;

$$\pi : \mathbf{M} \longrightarrow \bigoplus_{\lambda\in\mathcal{P}_{m|n}\cap\mathcal{P}_{u|v}} \mathbf{B}_{m|n}(\lambda)\times\mathbf{B}_{u|v}(\lambda).$$

Proof. For convenience, we assume that $m \geq u$ and $n \geq v$. Suppose that $A = (a_{bb'}) \in \mathbf{M}_{\rm h.w.}$ is given. We claim that $A = A_\lambda$ for some $\lambda \in \mathcal{P}_{m|n}\cap\mathcal{P}_{u|v}$ (see (3.8)). We may assume that A is a non-zero matrix since the zero matrix corresponds to $A_{(0)}$.

STEP 1. Suppose that $A = A(\mathbf{i},\mathbf{j}) = A(\mathbf{k},\mathbf{l})$ for $(\mathbf{i},\mathbf{j}) \in \Omega$, $(\mathbf{k},\mathbf{l}) \in \Omega^*$, and $\mathbf{i} = i_1\cdots i_r$ and $\mathbf{j} = j_1\cdots j_r$ for some $r \geq 1$. Let us write

$$\mathbf{i} = \mathbf{i}_{\overline{u}}\mathbf{i}_{\overline{u-1}}\cdots\mathbf{i}_{v-1}\mathbf{i}_v,$$

where $\mathbf{i}_b = i_{t_1}\cdots i_{t_b}$ ($b\in\mathbf{B}_{u|v}$, $t_b \geq 0$) is a subword of $\mathbf{i}$ such that $j_{t_1} = \cdots = j_{t_b} = b$. Similarly, we write $\mathbf{l} = \mathbf{l}_{\overline{m}}\cdots\mathbf{l}_n$.

Since $A \in \mathbf{M}_{\rm h.w.}$, $\pi(A) = (P(\mathbf{i}), P(\mathbf{l}))$ is a pair of highest weight tableaux. By Schensted's algorithm, we observe that the shape of $P(\mathbf{i}_{\overline{u}})$ is a single row, and all the letters in $\mathbf{i}_{\overline{u}}$ are placed in the first row of $P(\mathbf{i})$. Hence, we should have $\mathbf{i}_{\overline{u}} = \overline{m}\cdots\overline{m}$ ($a_{\overline{m}\,\overline{u}}$ times), equivalently, $a_{b\overline{u}} = 0$ for all $b \in \mathbf{B}_{m|n}\setminus\{\overline{m}\}$. By the same arguments, we have $\mathbf{l}_{\overline{m}} = \overline{u}\cdots\overline{u}$ ($a_{\overline{m}\,\overline{u}}$ times), or $a_{\overline{m}b} = 0$ for all $b\in\mathbf{B}_{u|v}\setminus\{\overline{u}\}$.

Next, consider $\mathbf{i}_{\overline{u-1}}$. Since $a_{\overline{m}\overline{u-1}} = 0$, all the letters in $\mathbf{i}_{\overline{u-1}}$ are greater than $\overline{m}$, and placed in the first two rows of $P(\mathbf{i})$. This implies that $\mathbf{i}_{\overline{u-1}} = \overline{m-1}\cdots\overline{m-1}$ ($a_{\overline{m-1}\,\overline{u-1}}$ times), and $a_{\overline{m}\,\overline{u}} \geq a_{\overline{m-1}\,\overline{u-1}}$ (see Figure 1). Similarly, $\mathbf{l}_{\overline{m-1}} = \overline{u-1}\cdots\overline{u-1}$ ($a_{\overline{m-1}\,\overline{u-1}}$ times). Repeating the above arguments, it follows that for $0\leq k\leq u-1$,

$$\mathbf{i}_{\overline{u-k}} = \underbrace{\overline{m-k}\cdots\overline{m-k}}_{a_{\overline{m-k}\,\overline{u-k}}}, \qquad \mathbf{l}_{\overline{m-k}} = \underbrace{\overline{u-k}\cdots\overline{u-k}}_{a_{\overline{m-k}\,\overline{u-k}}},$$

and

$$\lambda_1 \geq \cdots \geq \lambda_u,$$

where $\lambda_{k+1} = a_{\overline{m-k}\,\overline{u-k}}$ for $0 \leq k < u$.

STEP 2. Suppose that $a_{st} \neq 0$ for some $1 \leq s \leq n$ and $1\leq t\leq v$. Let ℓ be the maximum column index ($1\leq \ell\leq v$) such that $a_{s\ell}\neq 0$ for some $1\leq s\leq n$.

First, we claim that $a_{st} = 0$ for $s+t > \ell+1$.

Consider $a_{s\,\ell}$ for $1\leq s\leq n$. Suppose that $a_{s\,\ell}\neq 0$ for some $s\geq 2$. Then we have $e_{s-1}A\neq 0$, since $\epsilon^{(s-1)}(A) = (\cdots, +^{a_{s-1\,\ell}}, -^{a_{s\,\ell}})$ and there exists at least one $-$ in the $(s-1)$-signature of A. This is a contradiction. So, we have $a_{1\ell}\neq 0$ and $a_{s\ell} = 0$ for $s\geq 2$. Next, consider $a_{s\,\ell-1}$ for $1\leq s\leq n$. Suppose that $a_{s\,\ell-1}\neq 0$ for some $s\geq 3$. Then we also have $e_{s-1}A\neq 0$, since $\epsilon^{(s-1)}(A) = (\cdots, +^{a_{s-1\,\ell-1}}, -^{a_{s\,\ell-1}})$, which is a contradiction. Similarly, we can check that $a_{st} = 0$ for $1\leq t\leq \min(\ell, n)$ and $s > \ell - t + 1$.

Now, we claim that $a_{st} = a_{s't'}$ for $2\leq s+t = s'+t' \leq \ell+1$.

Consider a_{st} for $s+t = \ell+1$. Since $e_2A = 0$ and $e^*_{\ell-1}A = 0$, we have $a_{2\,\ell-1}\leq a_{1\,\ell}$ and $a_{2\,\ell-1}\geq a_{1\,\ell}$, respectively. Hence $a_{2\,\ell-1} = a_{1\,\ell}$. Continuing this argument, it follows that $a_{s\,\ell-s+1} = a_{1\ell}$ for $1\leq s\leq \min(\ell, n)$. Indeed, we have $\ell\leq n$. Otherwise, we have $a_{n\ \ell-n+1}\neq 0$, and $e^*_{\ell-n}A\neq 0$. Next, consider a_{st} for $s+t = \ell$. Since $e_1A = 0$

and

$$\epsilon^{(1)}(A) = (\cdots, -^{a_{2\,\ell-2}}, +^{a_{1\,\ell-1}}, -^{a_{2\,\ell-1}}, +^{a_{1\,\ell}})$$

(note that $a_{2\,\ell-1} = a_{1\,\ell}$), we have $a_{2\,\ell-2} \leq a_{1\,\ell-1}$. Since $e^*_{\ell-2}A = 0$ and

$$\epsilon^{(\ell-2)}(A) = (\cdots, -^{a_{1\,\ell-1}}, +^{a_{2\,\ell-2}}, -^{a_{2\,\ell-1}}, +^{a_{3\,\ell-2}})$$

(note that $a_{3\,\ell-2} = a_{2\,\ell-1} = a_{1\,\ell}$), we have $a_{2\,\ell-2} \geq a_{1\,\ell-1}$. Hence, $a_{2\,\ell-2} = a_{1\,\ell-1}$. Similarly, we can check that $a_{1\,\ell-1} = a_{2\,\ell-2} = a_{3\,\ell-3} = \cdots = a_{\ell-1\,1}$.

Applying the above arguments successively, we conclude that $a_{s\,t} = a_{s'\,t'}$ for $2 \leq s + t = s' + t' \leq \ell + 1$. We set $\nu_k = a_{s\,k-s+1}$ for $1 \leq k \leq \ell$.

Step 3. Let ℓ be the maximum column index ($1 \leq \ell \leq v$) given in Step 2. We assume that $\ell = 0$ if $a_{st} = 0$ for all $1 \leq s \leq n$ and $1 \leq t \leq v$.

We claim that $a_{\overline{k}t} = 1$ for $\overline{m-u} \leq \overline{k} \leq \overline{1}$ and $1 \leq t \leq \ell$. Let us use the induction on t. If $\ell = 0$, then it is clear. Suppose that $\ell > 0$. Consider $P_1 = P(\mathbf{i}_{\overline{u}} \cdots \mathbf{i}_{\overline{1}}\mathbf{i}_1)$. We have seen in Step 1 that $P(\mathbf{i}_{\overline{u}} \cdots \mathbf{i}_{\overline{1}})$ is an (m,n)-hook semistandard tableau whose shape is a partition $(\lambda_1, \cdots, \lambda_u)$ with the k^{th} row filled with $\overline{m-k+1}$ $(1 \leq k \leq u)$. When we insert the word $\mathbf{i}_1$ into $P(\mathbf{i}_{\overline{u}} \cdots \mathbf{i}_{\overline{1}})$, all the letters in $\mathbf{i}_1$ are placed in the first column of P_1. If $a_{\overline{k}\,1} = 0$ for some $\overline{m-u} \leq \overline{k} \leq \overline{1}$, then there exists at least one letter in $\mathbf{B}^-_{m|n}$ placed in the first m rows of P_1 and hence $P(\mathbf{i})$. This contradicts the fact that $P(\mathbf{i})$ is a highest weight tableau.

For $t < \ell$, suppose that $a_{\overline{k}t'} = 1$ for $\overline{m-u} \leq \overline{k} \leq \overline{1}$ and $1 \leq t' \leq t$. Put $P_t = P(\mathbf{i}_{\overline{u}} \cdots \mathbf{i}_{\overline{1}}\mathbf{i}_1 \cdots \mathbf{i}_t)$. Then, each k^{th} row of P_t $(1 \leq k \leq m)$ is filled with $\overline{m-k+1}$. If we cut out the first m rows of P_t, then the remaining tableau consists of exactly t columns. Moreover, if we read its k^{th} column $(1 \leq k \leq t)$ from top to bottom, then the associated word is given by

$$k^{\nu_k+\cdots+\nu_t}(k+1)^{\nu_{t+1}}(k+2)^{\nu_{t+2}} \cdots (\ell-t+k)^{\nu_\ell}.$$

Since

$$\mathbf{i}_{t+1} = (\overline{m-u})^{a_{\overline{m-u}\,t+1}} \cdots \overline{1}^{a_{\overline{1}\,t+1}} 1^{\nu_{t+1}} \cdots (\ell-t)^{\nu_\ell},$$

it is not difficult to see that for $1 \leq k \leq \ell - t$, ν_{t+k} $(t+k)$'s are bumped out of the t^{th} column and inserted into the $(t+1)^{\text{st}}$ column, when we insert the word $\mathbf{i}_{t+1}$ into P_t (note that $\nu_\ell > 0$ and $\mathbf{i}_{t+1}$ is not an empty word). So, if $a_{\overline{k}\,t+1} = 0$ for some $\overline{m-u} \leq \overline{k} \leq \overline{1}$, then at least one letter in $\mathbf{B}^-_{m|n}$ happens to be placed in the first m rows of P_{t+1} and hence $P(\mathbf{i})$, which is a contradiction. This completes the induction.

Step 4. Consider $a_{\overline{1}\,t}$ for $\ell < t \leq v$. Suppose that $a_{\overline{1}\,t} = 1$ and $a_{\overline{1}\,t-1} = 0$ for some $\ell < t \leq v$. Then, we have $e_{t-1}A \neq 0$ since $\epsilon^{(t-1)}(A) = (\cdots, +, -)$, which is a contradiction.

Next, consider $a_{\overline{2}\,t}$ for $\ell < t \leq v$. Suppose that $a_{\overline{2}\,t} = 1$ and $a_{\overline{2}\,t-1} = 0$ for some $\ell + 1 < t \leq v$. We assume that t is the minimum index such that $a_{\overline{2}\,t} = 1$ and $a_{\overline{2}\,t-1} = 0$. If $a_{\overline{1}\,t-1} = a_{\overline{1}\,t} = 0$ or $a_{\overline{1}\,t-1} = a_{\overline{1}\,t} = 1$, then $e^*_{t-1}A \neq 0$. If $a_{\overline{1}\,t-1} = 1$ and $a_{\overline{1}\,t} = 0$, then $a_{\overline{1}t'} = 1$ for $t' < t$, and $a_{\overline{1}t''} = 0$ for $t'' \geq t$, which implies that there exists a $\overline{1}$-good $-$ sign in A with respect to the $\mathfrak{gl}_{m|n}$-crystal structure, and $e_{\overline{1}}A \neq 0$. So, in any case, we get a contradiction. Moreover, we see that there is no t $(\ell < t \leq v)$ such that $a_{\overline{2}t} = 0$ and $a_{\overline{1}t} = 1$, since $e_{\overline{1}}A = 0$.

Now, applying the above arguments successively to $a_{\overline{k}\,t}$ for $1 < k \leq m-u$ and $\ell < t \leq v$, it follows that if $a_{\overline{k}t} = 0$ for $1 < k \leq m-u$ and $\ell < t \leq v$, then $a_{\overline{k}\,t+1} = a_{\overline{k-1}\,t} = 0$. This leads to

$$\lambda_{u+1} \geq \cdots \geq \lambda_m,$$

where $\lambda_{u+k} = \sum_{1\leq t\leq v} a_{\overline{m-u-k+1}\,t}$ $(1 \leq k \leq m-u)$. Also, we have $\lambda_u \geq \lambda_{u+1}$ since $e_{\overline{m-u}}A = 0$.

Let $(\lambda_{m+1}, \lambda_{m+2}, \cdots, \lambda_r)$ be the partition determined by (3.7), and put

$$\lambda = (\lambda_1, \cdots, \lambda_m, \lambda_{m+1}, \cdots, \lambda_r).$$

Note that $\lambda_{m+1} = \ell$, where ℓ is the maximum column index given in Step 2, and $\lambda_m \geq \lambda_{m+1}$ by Step 3. So, λ is a Young diagram, and $\lambda \in \mathcal{P}_{m|n} \cap \mathcal{P}_{u|v}$ by construction. Finally, we conclude that $A = A_\lambda$ given in (3.8). □

Let $x = \{\, x_b \,|\, b \in \mathbf{B}_{m|n} \,\}$ and $y = \{\, y_{b'} \,|\, b' \in \mathbf{B}_{u|v} \,\}$. The character of $\mathbf{M}$ is given by

$$\mathrm{ch}\mathbf{M} = \sum_{A\in\mathbf{M}} x^{\mathrm{wt}(A)} y^{\mathrm{wt}^*(A)} = \frac{\prod_{|b|\neq|b'|}(1 + x_b y_{b'})}{\prod_{|b|=|b'|}(1 - x_b y_{b'})},$$

where $b \in \mathbf{B}_{m|n}$ and $b' \in \mathbf{B}_{u|v}$. By Theorem 3.11, we recover the *super Cauchy identity*;

$$\frac{\prod_{|b|\neq|b'|}(1 + x_b y_{b'})}{\prod_{|b|=|b'|}(1 - x_b y_{b'})} = \sum_{\lambda\in\mathcal{P}_{m|n}\cap\mathcal{P}_{u|v}} hs_\lambda(x) hs_\lambda(y). \tag{3.9}$$

Remark 3.12. Let $\mathcal{A}_n = \{\, a_1 < \cdots < a_n \,\}$ be the set of n letters with a linear ordering. For a partition λ with length at most n, a tableau T obtained by filling λ with the letters in $\mathcal{A}_n$ is called a *semistandard tableau of shape* λ if the entries in each row (resp. column) are weakly (resp. strictly) increasing from left to right (resp. from top to bottom). We denote by $\mathcal{SST}_n(\lambda)$ the set of all semistandard tableau of shape λ with entries in $\mathcal{A}_n$. For example, $\mathbf{B}_{m|0}(\lambda)$ $(\lambda \in \mathcal{P}_{m|0})$ may be identified with $\mathcal{SST}_m(\lambda)$, and $\mathbf{B}_{0|n}(\lambda)$ $(\lambda \in \mathcal{P}_{0|n})$ with $\mathcal{SST}_n(\lambda')$, where λ' is the conjugate of λ.

With this notation, we can recover several variations of the original Knuth correspondence from Theorem 3.11 (cf.[5]). If we put $n = v = 0$ or $m = u = 0$, then we have two kinds of Knuth correspondence, where the one is given by the column insertion of words and the other is given by the row insertion of words. If we put $m = v = 0$ or $n = u = 0$, then we obtain the dual Knuth correspondence. Similarly, one may also obtain other variations from the decomposition given in next section (see Theorem 4.5).

3.4. **Diagonal action on symmetric matrices.** Suppose that $m = u$ and $n = v$. Set

$$\mathfrak{M} = \{\, A \in \mathbf{M} \,|\, A = A^t \,\}, \tag{3.10}$$

the set of all symmetric matrices in $\mathbf{M}$. Let us consider the diagonal action of $\mathfrak{gl}_{m|n} \oplus \mathfrak{gl}_{m|n}$ on $\mathfrak{M}$. That is, for $i \in I_{m|n}$ and $A \in \mathfrak{M}$, we define

$$\mathbf{e}_i A = e_i e_i^* A = e_i^* e_i A,$$
$$\mathbf{f}_i A = f_i f_i^* A = f_i^* f_i A.$$

Note that $P_1(A) = P_2(A)$ for $A \in \mathfrak{M}$ (see (3.5)). Put $\mathrm{wt}(A) = \mathrm{wt}(P_1(A)) = \mathrm{wt}(P_2(A))$, $\varepsilon_i(A) = \max\{\, k \,|\, \mathbf{e}_i^k A \neq 0 \,\}$, and $\varphi_i(A) = \max\{\, k \,|\, \mathbf{f}_i^k A \neq 0 \,\}$ for $i \in I_{m|n}$ and $A \in \mathfrak{M}$. Then we have

Proposition 3.13. *$\mathfrak{M}$ is a crystal graph for $\mathfrak{gl}_{m|n}$, which decomposes as follows;*

$$\mathfrak{M} \simeq \bigoplus_{\lambda \in \mathcal{P}_{m|n}} \mathbf{B}_{m|n}(\lambda).$$

Proof. If $\mathbf{e}_i A \neq 0$ or $\mathbf{f}_i A \neq 0$ for $A \in \mathfrak{M}$ and $i \in I_{m|n}$, then

$$(\mathbf{e}_i A)^t = (e_i e_i^* A)^t = e_i^* e_i A^t = \mathbf{e}_i A \in \mathfrak{M},$$
$$(\mathbf{f}_i A)^t = (f_i f_i^* A)^t = f_i^* f_i A^t = \mathbf{f}_i A \in \mathfrak{M}.$$

Hence, $\mathbf{e}_i, \mathbf{f}_i : \mathfrak{M} \to \mathfrak{M} \cup \{0\}$ are well-defined operators for $i \in I_{m|n}$.

For $A \in \mathfrak{M}$ and $i \in I_{m|n}$, we have $x_i A \neq 0$ if and only if $x_i^* A \neq 0$ $(x = e, f)$ since A is symmetric. Hence, by Lemma 3.6, we have

$$\mathbf{e}_i A \neq 0 \Leftrightarrow P_1(e_i A) \neq 0 \Leftrightarrow e_i P_1(A) \neq 0,$$
$$\mathbf{f}_i A \neq 0 \Leftrightarrow P_1(f_i A) \neq 0 \Leftrightarrow f_i P_1(A) \neq 0,$$

for $i \in I_{m|n}$. This implies that $\mathfrak{M}$ is a $\mathfrak{gl}_{m|n}$-crystal.

Next, consider the decomposition of $\mathfrak{M}$. For $A \in \mathfrak{M}$, A is $\mathfrak{gl}_{m|n}$-equivalent to $P_1(A)$. So each connected component in $\mathfrak{M}$ is generated by A_λ for some $\lambda \in \mathcal{P}_{m|n}$ by Theorem 3.11. Since $A_\lambda \in \mathfrak{M}$ for $\lambda \in \mathcal{P}_{m|n}$, the set of all the highest weight elements in $\mathfrak{M}$ is equal to $\mathbf{M}_{\text{h.w.}}$. □

Now, let us show that there is an interesting relation between the diagonal entries of a matrix in $\mathfrak{M}$ and the shape of the corresponding tableau (cf.[14]), and hence obtain a family of subcrystals of $\mathfrak{M}$, which also have nice decompositions. For $A = (a_{bb'})_{b,b' \in \mathbf{B}_{m|n}} \in \mathfrak{M}$, let

$$\mathfrak{o}(A) = \Big|\{\, b \in \mathbf{B}^+_{m|n} \,|\, a_{bb} \equiv 1 \pmod 2 \,\}\Big| + \sum_{b \in \mathbf{B}^-_{m|n}} a_{bb}. \tag{3.11}$$

Proposition 3.14. *Let $\mathfrak{M}_k = \{\, A \in \mathfrak{M} \,|\, \mathfrak{o}(A) = k \,\}$ for $k \geq 0$. Then $\mathfrak{M}_k$ is a subcrystal of $\mathfrak{M}$, and decomposes as follows;*

$$\mathfrak{M}_k \simeq \bigoplus_{\substack{\lambda \in \mathcal{P}_{m|n} \\ o(\lambda) = k}} \mathbf{B}_{m|n}(\lambda),$$

where $o(\lambda)$ is the number of odd parts in λ.

Proof. Fix $k \geq 0$. First, we will check that $\mathfrak{M}_k$ together with 0 is stable under $\mathbf{e}_i, \mathbf{f}_i$ $(i \in I_{m|n})$, which implies that $\mathfrak{M}_k$ is a subcrystal of $\mathfrak{M}$.

Given $A = (a_{bb'}) \in \mathfrak{M}_k$ and $i \in I_{m|n}$, suppose that $\mathbf{f}_i A \neq 0$. We assume that the diagonal entries of A are changed under $\mathbf{f}_i$, equivalently under f_i or f_i^*. Otherwise, it is clear that $\mathbf{f}_i A \in \mathfrak{M}_k$. For convenience, we write

$$A[i] = \begin{pmatrix} a_{bb} & a_{bb'} \\ a_{b'b} & a_{b'b'} \end{pmatrix},$$

where $b < b'$ are the indices such that $\langle h_i, \epsilon_b \rangle, \langle h_i, \epsilon_{b'} \rangle \neq 0$.

Case 1. $i=\overline{k}$, $(1\le k\le m-1)$. Consider

$$A[i]=\begin{pmatrix} a_{\overline{k+1}\,\overline{k+1}} & a_{\overline{k+1}\,\overline{k}} \\ a_{\overline{k}\,\overline{k+1}} & a_{\overline{k}\,\overline{k}} \end{pmatrix}=\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Suppose that $(f_iA)[i]=\begin{pmatrix} a-1 & b \\ b+1 & c \end{pmatrix}$. This implies that with respect to f_i, we have $\epsilon^{(i)}(A)=(\cdots,-^b,+^a,-^c,+^b,\cdots)$, where $a>c$ and the i-good $+$ sign of A appears in $+^a$ of $\epsilon^{(i)}(A)$. Note that $\epsilon^{(i)}(f_iA)=(\cdots,-^b,+^{a-1},-^c,+^{b+1},\cdots)$ with respect to f_i^*.

If $a\equiv c \pmod 2$, then we still have $a-1>c$, and the i-good $+$ sign of f_iA with respect to f_i^* appears in $+^{a-1}$, and we have

$$(\mathbf{f}_iA)[i]=(f_i^*f_iA)[i]=\begin{pmatrix} a-2 & b+1 \\ b+1 & c \end{pmatrix}.$$

If $a\not\equiv c \pmod 2$, then we have

$$(\mathbf{f}_iA)[i]=(f_i^*f_iA)[i]=\begin{pmatrix} a-2 & b+1 \\ b+1 & c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a-1 & b \\ b & c+1 \end{pmatrix}.$$

In any case, we have $\mathfrak{o}(\mathbf{f}_iA)=k$ and $\mathbf{f}_iA\in\mathfrak{M}_k$.

Next, suppose that $(f_iA)[i]=\begin{pmatrix} a & b-1 \\ b & c+1 \end{pmatrix}$. Then we must have $(\mathbf{f}_iA)[i]=\begin{pmatrix} a & b-1 \\ b-1 & c+2 \end{pmatrix}$, and $\mathfrak{o}(\mathbf{f}_iA)=k$.

Case 2. $i=k$, $(1\le k\le n-1)$. Consider

$$A[i]=\begin{pmatrix} a_{k\,k} & a_{k\,k+1} \\ a_{k+1\,k} & a_{k+1\,k+1} \end{pmatrix}=\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Suppose that $a>0$ and $(f_iA)[i]=\begin{pmatrix} a-1 & b \\ b+1 & c \end{pmatrix}$. Then with respect to f_i, we have $\epsilon^{(i)}(A)=(\cdots,+^a,-^b,+^b,-^c,\cdots)$, where the i-good $+$ sign of A appears in $+^a$. Since $\epsilon^{(i)}(f_iA)=(\cdots,+^{a-1},-^b,+^{b+1},-^c,\cdots)$ with respect to f_i^*, the i-good $+$ sign of f_iA appears in $+^{b+1}$, and we have

$$(\mathbf{f}_iA)[i]=(f_i^*f_iA)[i]=\begin{pmatrix} a-1 & b \\ b & c+1 \end{pmatrix},$$

which implies that $\mathfrak{o}(\mathbf{f}_iA)=k$.

Next, suppose that $b>0$ and $(f_iA)[i]=\begin{pmatrix} a & b-1 \\ b & c+1 \end{pmatrix}$. But, this can't happen since $\epsilon^{(i)}(A)=(\cdots,+^a,-^b,+^b,-^c,\cdots)$ with respect to f_i, and the pair $(-^b,+^b)$ cancels out.

Case 3. $i=0$. Consider

$$A[0]=\begin{pmatrix} a_{\overline{1}\,\overline{1}} & a_{\overline{1}\,1} \\ a_{1\,\overline{1}} & a_{1\,1} \end{pmatrix}=\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Then we have

$$(\mathbf{f}_0 A)[0] = (f_0^* f_0 A)[0] = \begin{pmatrix} a-2 & b+1 \\ b+1 & c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a-1 & b \\ b & c+1 \end{pmatrix},$$

(it can't happen that $(\mathbf{f}_0 A)[0] = \begin{pmatrix} a & b-1 \\ b-1 & c+2 \end{pmatrix}$). In any case, we have $\mathfrak{o}(\mathbf{f}_i A) = k$.

Similarly, we can check that $\mathbf{e}_i \mathfrak{M}_k \subset \mathfrak{M}_k \cup \{0\}$ for $i \in I_{m|n}$. Therefore, $\mathfrak{M}_k$ is a crystal graph for $\mathfrak{gl}_{m|n}$.

Next, we observe that for $\lambda = (\lambda_k)_{k\geq 1} \in \mathcal{P}_{m|n}$,

$$\mathfrak{o}(A_\lambda) = |\{\, i \,|\, 1 \leq i \leq m,\, \lambda_i \text{ is odd} \,\}| + \sum_{k\geq 0} \nu_{2k+1},$$

(see (3.8)). Since ν_i is the number of occurrences of i in $(\lambda_{m+1}, \lambda_{m+2}, \cdots)$ (see (3.7)), it follows that $\mathfrak{o}(A_\lambda)$ is the number of odd parts in λ, say $o(\lambda)$, and hence $A_\lambda \in \mathfrak{M}_k$ if and only if $o(\lambda) = k$. □

Corollary 3.15. *Under the above hypothesis, we have* $\mathfrak{M} = \bigoplus_{k\geq 0} \mathfrak{M}_k$, *and in particular,*

$$\mathfrak{M}_0 \simeq \bigoplus_{\substack{\lambda \in \mathcal{P}_{m|n} \\ \lambda\,:\,even}} \mathbf{B}_{m|n}(\lambda).$$

Proof. It follows from the fact that $\mathfrak{o}(A_\lambda) = 0$ if and only if λ is even (that is, each part of λ is even). □

Remark 3.16. (1) A special case of Proposition 3.14 was first observed in [14]. Let us give a brief explanation. Put $m = 0$ in Corollary 3.15. We identify $\mathbf{B}_{0|n}$ with $\mathcal{A}_n$ (see Remark 3.12), and $\mathbf{B}_{0|n}(\lambda)$ with $\mathcal{SST}_n(\lambda')$. Hence, the set of all $n \times n$ symmetric matrices of non-negative integers with $\mathrm{tr}(A) = k$, is in one-to-one correspondence with $\bigsqcup_{o(\lambda')=k} \mathcal{SST}_n(\lambda)$ (Theorem 4 [14]) where the union is given over all partitions with the number of odd columns k.

(2) If we consider the characters of the decompositions in Proposition 3.13 and Corollary 3.15, then we obtain the following identities (cf.[15]);

$$\frac{\prod_{b<b',\,|b|\neq|b'|}(1+x_b x_{b'})}{\prod(1-x_b)\prod_{b<b',\,|b|=|b'|}(1-x_b x_{b'})} = \sum_{\lambda\in\mathcal{P}_{m|n}} hs_\lambda(x),$$

where $b, b' \in \mathbf{B}_{m|n}$,

$$\frac{\prod_{b<b',\,|b|\neq|b'|}(1+x_b x_{b'})}{\prod_{|b|=0}(1-x_b^2)\prod_{b<b',\,|b|=|b'|}(1-x_b x_{b'})} = \sum_{\substack{\lambda\in\mathcal{P}_{m|n} \\ \lambda\,:\,\text{even}}} hs_\lambda(x),$$

where $b, b' \in \mathbf{B}_{m|n}$.

4. Dual construction

In this section, we discuss a bicrystal graph associated to the super exterior algebra $\Lambda(\mathbb{C}^{m|n} \otimes \mathbb{C}^{u|v})$, and its explicit decomposition.

Suppose that m, n, u, v are non-negative integers such that $m+n, u+v > 0$. We set

$$\mathbf{M}^\sharp_{m|n,u|v} = \{\, A = (a_{bb'})_{b\in \mathbf{B}_{m|n}, b'\in \mathbf{B}_{u|v}} \mid \text{(1) } a_{bb'} \in \mathbb{Z}_{\geq 0},\ \text{(2) } a_{bb'} \leq 1 \text{ if } |b| = |b'| \,\}. \tag{4.1}$$

For convenience, we write $\mathbf{M}^\sharp = \mathbf{M}^\sharp_{m|n,u|v}$.

As in the case of $\mathbf{M}$, we identify a matrix in $\mathbf{M}^\sharp$ with a biword given by reading the row and column indices of non-zero entries of the matrix with respect to a linear ordering. First, we let

$$\begin{aligned}\Omega^\sharp_{m|n,u|v} = \Omega^\sharp = \{\, (\mathbf{i},\mathbf{j}) \in \mathcal{W}_{m|n} \times \mathcal{W}_{u|v} \mid \ & \text{(1) } \mathbf{i} = i_1\cdots i_r \text{ and } \mathbf{j} = j_1 \cdots j_r \text{ for some } r \geq 0, \\ & \text{(2) } (i_1, j_1) \preceq \cdots \preceq (i_r, j_r), \\ & \text{(3) } |i_k| = |j_k| \text{ implies } (i_k, j_k) \neq (i_{k\pm 1}, j_{k\pm 1}), \},\end{aligned} \tag{4.2}$$

where for (i,j) and $(k,l) \in \mathbf{B}_{m|n} \times \mathbf{B}_{u|v}$, the linear ordering $\prec$ is given by

$$(i,j) \prec (k,l) \quad \Leftrightarrow \quad \begin{cases} (j < l) & \text{or,} \\ (j = l \in \mathbf{B}^+_{u|v}, \text{ and } i < k) & \text{or,} \\ (j = l \in \mathbf{B}^-_{u|v}, \text{ and } i > k) & . \end{cases} \tag{4.3}$$

We define $e_i, f_i : \Omega^\sharp \to \Omega^\sharp \cup \{0\}$ $(i \in I_{m|n})$ by

$$e_i(\mathbf{i},\mathbf{j}) = (e_i\mathbf{i},\mathbf{j}), \qquad f_i(\mathbf{i},\mathbf{j}) = (f_i\mathbf{i},\mathbf{j}),$$

for $(\mathbf{i},\mathbf{j}) \in \Omega^\sharp$. Set $\mathrm{wt}(\mathbf{i},\mathbf{j}) = \mathrm{wt}(\mathbf{i})$, $\varepsilon_i(\mathbf{i},\mathbf{j}) = \varepsilon_i(\mathbf{i})$ and $\varphi_i(\mathbf{i},\mathbf{j}) = \varphi_i(\mathbf{i})$ $(i \in I_{m|n})$. Then it is easy to see that $\Omega^\sharp$ is a crystal graph for $\mathfrak{gl}_{m|n}$ (cf. Lemma 3.1).

For $(\mathbf{i},\mathbf{j}) \in \Omega^\sharp_{m|n,u|v}$, we define $A(\mathbf{i},\mathbf{j}) = (a_{bb'})$ to be a matrix in $\mathbf{M}^\sharp$, where $a_{bb'}$ is the number of k's such that $(i_k, j_k) = (b, b')$ for $b \in \mathbf{B}_{m|n}$ and $b' \in \mathbf{B}_{u|v}$. Then, the map $(\mathbf{i},\mathbf{j}) \mapsto A(\mathbf{i},\mathbf{j})$ is a bijection between $\Omega^\sharp_{m|n,u|v}$ and $\mathbf{M}^\sharp$. Hence, $\mathbf{M}^\sharp$ is a crystal graph for $\mathfrak{gl}_{m|n}$ with this identification.

Next, we introduce $(\Omega^\sharp_{m|n,u|v})^*$ to define a $\mathfrak{gl}_{u|v}$-crystal structure on $\mathbf{M}^\sharp$. Recall that in Section 3, a biword in Ω^* was obtained by reading the row and column indices of the non-zero entries in the transpose of a given matrix with respect to the same lexicographic ordering used in Ω. But in the case of $\mathbf{M}^\sharp$, we need another linear ordering. That is, we set

$$\begin{aligned}(\Omega^\sharp_{m|n,u|v})^* = (\Omega^\sharp)^* = \{\, (\mathbf{k},\mathbf{l}) \in \mathcal{W}_{m|n} \times \mathcal{W}_{u|v} \mid \ & \text{(1) } \mathbf{k} = k_1\cdots k_r \text{ and } \mathbf{l} = l_1 \cdots l_r \text{ for some } r \geq 0, \\ & \text{(2) } (k_1, l_1) \preceq' \cdots \preceq' (k_r, l_r), \\ & \text{(3) } |k_t| = |l_t| \text{ implies } (k_t, l_t) \neq (k_{t\pm 1}, l_{t\pm 1}), \},\end{aligned} \tag{4.4}$$

where for (i,j) and $(k,l) \in \mathbf{B}_{m|n} \times \mathbf{B}_{u|v}$, the linear ordering $\prec'$ is given by

$$(i,j) \prec' (k,l) \quad \Leftrightarrow \quad \begin{cases} (i > k) & \text{or,} \\ (i = k \in \mathbf{B}^+_{u|v}, \text{ and } j < l) & \text{or,} \\ (i = k \in \mathbf{B}^-_{u|v}, \text{ and } j > l) & . \end{cases} \tag{4.5}$$

Clearly, we have a bijection $(\mathbf{k},\mathbf{l}) \mapsto A(\mathbf{k},\mathbf{l})$ from $(\Omega^\sharp)^*$ to $\mathbf{M}^\sharp$.

Similarly, we define $e_j^*, f_j^* : (\Omega^\sharp)^* \to (\Omega^\sharp)^* \cup \{0\}$ $(j \in I_{u|v})$ by

$$e_j^*(\mathbf{k},\mathbf{l}) = (\mathbf{k}, e_j\mathbf{l}), \qquad f_j^*(\mathbf{k},\mathbf{l}) = (\mathbf{k}, f_j\mathbf{l}),$$

for $(\mathbf{k},\mathbf{l}) \in (\Omega^\sharp)^*$. Set $\mathrm{wt}^*(\mathbf{k},\mathbf{l}) = \mathrm{wt}(\mathbf{l})$, $\varepsilon_j^*(\mathbf{k},\mathbf{l}) = \varepsilon_j(\mathbf{l})$ and $\varphi_j^*(\mathbf{k},\mathbf{l}) = \varphi_j(\mathbf{l})$ $(j \in I_{u|v})$. Then $(\Omega^\sharp)^*$ is a crystal graph for $\mathfrak{gl}_{u|v}$ (cf. Lemma 3.1), and hence so is $\mathbf{M}^\sharp$.

Example 4.1. Suppose that $m|n = u|v = 2|2$ and

$$A = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right) \in \mathbf{M}^\sharp.$$

Then $A = A(\mathbf{i},\mathbf{j}) = A(\mathbf{k},\mathbf{l})$ for $(\mathbf{i},\mathbf{j}) \in \Omega^\sharp$ and $(\mathbf{k},\mathbf{l}) \in (\Omega^\sharp)^*$, where

$$\begin{aligned} \mathbf{i} &= \overline{2}\ 2\ 2\ \overline{2}\ 1\ 1\ \overline{1}\ \overline{1}\ \overline{1}, \quad & \mathbf{k} &= 2\ 2\ 1\ 1\ \overline{1}\ \overline{1}\ \overline{1}\ \overline{2}\ \overline{2}, \\ \mathbf{j} &= \overline{2}\ \overline{2}\ \overline{2}\ \overline{1}\ \overline{1}\ 1\ 1\ 1\ 2, \quad & \mathbf{l} &= \overline{2}\ \overline{2}\ 1\ \overline{1}\ 1\ 1\ 2\ \overline{2}\ \overline{1}. \end{aligned}$$

For $A \in \mathbf{M}^\sharp$, we have $A = A(\mathbf{i},\mathbf{j}) = A(\mathbf{k},\mathbf{l})$ for unique $(\mathbf{i},\mathbf{j}) \in \Omega^\sharp$ and $(\mathbf{k},\mathbf{l}) \in (\Omega^\sharp)^*$. Then we define

$$\pi^\sharp(A) = (P_1^\sharp(A), P_2^\sharp(A)) = (P(\mathbf{i}), P(\mathbf{l})). \tag{4.6}$$

By definition, we have $A \simeq_{\mathfrak{gl}_{m|n}} P_1^\sharp(A)$, and $A \simeq_{\mathfrak{gl}_{u|v}} P_2^\sharp(A)$.

Now, we have following analogue of Lemma 3.4 and Proposition 3.7;

Proposition 4.2. $\mathbf{M}^\sharp$ *is a* $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$*-bicrystal, and for each connected component* $\mathbf{C}$ *in* $\mathbf{M}^\sharp$*,* $\pi^\sharp$ *gives the following isomorphism of* $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$*-bicrystals;*

$$\pi^\sharp : \mathbf{C} \longrightarrow \mathbf{B}_{m|n}(\lambda) \times \mathbf{B}_{u|v}(\mu)$$

for some $\lambda \in \mathcal{P}_{m|n}$ *and* $\mu \in \mathcal{P}_{u|v}$.

Proof. For $A \in \mathbf{M}^\sharp$, let A^ρ be the matrix given as a clock-wise rotation of A by 90° (see Figure 2). The map $A \mapsto A^\rho$ gives a one-to-one correspondence from $\mathbf{M}^\sharp_{m|n,u|v}$ to $\mathbf{M}_{u|v,n|m}$. Moreover, we have

$$\begin{aligned} &(e_{\overline{k}}A)^\rho = f_k^* A^\rho, \quad (f_{\overline{k}}A)^\rho = e_k^* A^\rho, \quad (1 \le k \le m-1), \\ &(e_l A)^\rho = f_{\overline{l}}^* A^\rho, \quad (f_l A)^\rho = e_{\overline{l}}^* A^\rho, \quad (1 \le l \le n-1), \\ &(e_0 A)^\rho = f_0^* A^\rho, \quad (f_0 A)^\rho = e_0^* A^\rho, \end{aligned}$$

and $(x_j^* A)^\rho = x_j A^\rho$ for $j \in I_{u|v}$ and $x = e, f$. By Lemma 3.4, e_i, f_i $(i \in I_{m|n})$ commute with e_j^*, f_j^* $(j \in I_{u|v})$ on $\mathbf{M}^\sharp$. Hence, $\mathbf{M}^\sharp$ is a $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$-bicrystal.

Applying the same arguments in Lemma 3.5, 3.6 and Proposition 3.7, we conclude that each connected component of $\mathbf{M}^\sharp$ is isomorphic to $\mathbf{B}_{m|n}(\lambda) \times \mathbf{B}_{u|v}(\mu)$ for some $\lambda \in \mathcal{P}_{m|n}$ and $\mu \in \mathcal{P}_{u|v}$. $\square$

Example 4.3. Let A be the matrix given in Example 4.1. Then we have

$$\pi^\sharp(A) = \left(\begin{array}{ccccc} & & & & \\ \overline{2} & \overline{2} & \overline{1} & 1 & 2 \\ \overline{1} & \overline{1} & & & \\ 1 & 2 & & & \\ & & & & \end{array}, \begin{array}{ccc} \overline{2} & \overline{2} & \overline{2} \\ \overline{1} & \overline{1} & \overline{1} \\ 1 & & \\ 1 & & \\ 2 & & \end{array} \right).$$

$$
\left(\begin{array}{ccc|ccc}
0 & 1 & 1 & 0 & 0 & 2 \\
1 & 0 & 0 & 2 & 1 & 0 \\
\hline
3 & 0 & 1 & 1 & 0 & 1 \\
0 & 2 & 0 & 0 & 0 & 0
\end{array}\right)^{\rho}
=
\left(\begin{array}{cc|cc}
0 & 3 & 1 & 0 \\
2 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
\hline
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 2
\end{array}\right)
$$

FIGURE 2. A clock-wise rotation of A by 90°

Set

$$\mathbf{M}^{\sharp}_{\text{h.w.}} = \{\, A \in \mathbf{M}^{\sharp} \,|\, \pi^{\sharp}(A) = (H^{\lambda}_{m|n}, H^{\mu}_{u|v}) \text{ for some } (\lambda,\mu) \in \mathcal{P}_{m|n} \times \mathcal{P}_{u|v} \,\}, \tag{4.7}$$

which is the set of all the highest weight elements in $\mathbf{M}^{\sharp}$.

Suppose that $\lambda = (\lambda_k)_{k\geq 1} \in \mathcal{P}_{m|n} \cap \mathcal{P}_{v|u}$ is given, and let $\lambda' = (\lambda'_k)_{k\geq 1}$ be the conjugate of λ. We define $A^{\sharp}_{\lambda} = (a^{\sharp}_{bb'}) \in \mathbf{M}^{\sharp}$ by

(1) for $0 \leq k < m$, $0 \leq l < u$,
$$a^{\sharp}_{\overline{m-k}\,\overline{u-l}} = \begin{cases} 1, & \text{if } \lambda_{k+1} \geq l+1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) for $0 \leq k < m$, $1 \leq t \leq v$,
$$a^{\sharp}_{\overline{m-k}\,t} = \begin{cases} \mu_t, & \text{if } k+1 = t, \\ 0, & \text{otherwise,} \end{cases}$$
where $\mu_t = \max(\lambda_t - u, 0)$ for $t \geq 1$.

(3) for $1 \leq s \leq n$, $0 \leq l < u$,
$$a^{\sharp}_{s\,\overline{u-l}} = \begin{cases} \nu_{s+l}, & \text{if } 1 \leq s+l \leq \ell, \\ 0, & \text{otherwise,} \end{cases}$$
where $\nu_1, \cdots, \nu_\ell$ are given in (3.7).

(4) for $1 \leq s \leq n$, $1 \leq t \leq v$, $a^{\sharp}_{st} = 0$.

Note that $A^{\sharp}_{\lambda}$ is of the following form;

$$
A^{\sharp}_{\lambda} =
\left(\begin{array}{cccccc|cccc}
1 & 1 & & \cdots & & 1 & \mu_1 & & & \\
1 & 1 & & \cdots & & 1 & & \mu_2 & & \\
\vdots & \vdots & & & & \vdots & & & \mu_3 & \\
1 & 1 & & \cdots & & 1 & & & & \ddots \\
\vdots & & & & & & & & & \\
1 & 1 & \cdots & & 1 & & & & & \\
\hline
\nu_1 & \nu_2 & \cdots & \nu_\ell & & & & & & \\
\nu_2 & \nu_3 & \cdots & & & & & & & \\
\vdots & & & & & & & & & \\
\nu_\ell & & & & & & & & &
\end{array}\right). \tag{4.8}
$$

By Schensted's algorithm, we can check that

$$\pi^\sharp(A^\sharp_\lambda) = (H^\lambda_{m|n}, H^{\lambda'}_{u|v}).$$

Example 4.4. Suppose that $m|n = 3|2$ and $u|v = 4|3$. Let $\lambda = (7,6,2,2,1,1)$. Then

$$A^\sharp_\lambda = \left(\begin{array}{cccc|ccc} 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

and

$$\pi^\sharp(A^\sharp_\lambda) = \left(\begin{array}{ccccccc} \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} \\ \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \\ \overline{1} & \overline{1} & & & & & \\ 1 & 2 & & & & & \\ 1 & & & & & & \\ 1 & & & & & & \end{array} , \begin{array}{cccccc} \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} \\ \overline{3} & \overline{3} & \overline{3} & \overline{3} & & \\ \overline{2} & \overline{2} & & & & \\ \overline{1} & \overline{1} & & & & \\ 1 & 2 & & & & \\ 1 & 2 & & & & \\ 1 & & & & & \end{array} \right).$$

Now, we can characterize all the highest weight elements in $\mathbf{M}^\sharp$. The proof is almost the same as in Theorem 3.11.

Theorem 4.5. *We have*

$$\mathbf{M}^\sharp_{\text{h.w.}} = \{\, A^\sharp_\lambda \mid \lambda \in \mathcal{P}_{m|n} \cap \mathcal{P}_{v|u} \,\},$$

and hence the following isomorphism of $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{u|v})$-bicrystals;

$$\pi^\sharp : \mathbf{M}^\sharp \longrightarrow \bigoplus_{\lambda \in \mathcal{P}_{m|n} \cap \mathcal{P}_{v|u}} \mathbf{B}_{m|n}(\lambda) \times \mathbf{B}_{u|v}(\lambda').$$

$\square$

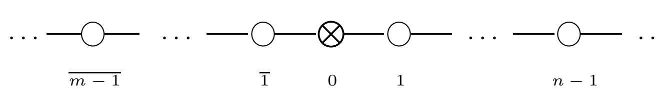
In terms of characters, we also recover the *dual Cauchy identity of hook Schur functions*;

$$\frac{\prod_{|b|=|b'|}(1 + x_b y_{b'})}{\prod_{|b|\neq|b'|}(1 - x_b y_{b'})} = \sum_{\lambda \in \mathcal{P}_{m|n} \cap \mathcal{P}_{v|u}} hs_\lambda(x) hs_{\lambda'}(y), \tag{4.9}$$

where $b \in \mathbf{B}_{m|n}$ and $b' \in \mathbf{B}_{u|v}$.

5. Semi-infinite construction

Let $\mathfrak{g}$ be a contragredinet Lie superalgebra of infinite rank whose associated Dynkin diagram is given by



(see [8]). Then for all $m, n \geq 1$, there is a natural embedding of $\mathfrak{gl}_{m|n}$ into $\mathfrak{g}$. Note that $\mathfrak{g}$ is not equal to $\mathfrak{gl}_{\infty|\infty}$ in the sense of [9], but a proper subalgebra of it. In this section, we study crystal graphs for $\mathfrak{g}$ which are generated by highest weight elements. Since a tensor power of the $\mathfrak{g}$-crystal associated to the natural representation does not have a

highest weight element, we introduce a $\mathfrak{g}$-crystal $\mathscr{F}$ consisting of semi-infinite words, which is analogous to the crystal graph for $\mathfrak{gl}_\infty$ associated to a level one fermionic Fock space representation (cf.[7]). We will show that each connected component of a tensor power of $\mathscr{F}$ can be realized as the set of semi-infinite semistandard tableaux, which is generated by a highest weight vector. Then by using the methods developed in the previous sections, we give an explicit decomposition of $\mathscr{F}^{\otimes u}$ $(u \geq 2)$ as a $(\mathfrak{g}, \mathfrak{gl}_u)$-bicrystal.

5.1. **Crystal graphs of semi-infinite words.** We may naturally define a crystal graph for $\mathfrak{g}$ by taking m and n to infinity in Definition 2.1. Set

$$\mathbf{B} = \{ \cdots < \overline{m} < \cdots < \overline{1} < 1 < \cdots < n < \cdots \},$$

and $\mathbf{B}^+$ (resp. $\mathbf{B}^-$) denotes the set of elements with degree 0 (resp. 1) in $\mathbf{B}$. The index set is given by

$$I = \{ \cdots , \overline{m}, \cdots , \overline{1}, 0, 1, \cdots , n, \cdots \}.$$

Then the simple root α_i and the simple coroot h_i $(i \in I)$ are defined in the same way. But, instead of $\bigoplus_{b\in\mathbf{B}} \mathbb{Z}\epsilon_b$, we use

$$P = \mathbb{Z}\Lambda \oplus \bigoplus_{b\in\mathbf{B}} \mathbb{Z}\epsilon_b$$

as the weight lattice of $\mathfrak{g}$, where $\Lambda = \Lambda_0 \in (\bigoplus_{i\in I} \mathbb{Z}h_i)^*$ is the fundamental weight such that $\langle h_i, \Lambda\rangle = \delta_{0i}$ for $i \in I$. Note that $\mathbf{B}$ is the crystal graph for $\mathfrak{g}$ associated to the natural representation of $\mathfrak{g}$.

Now, we define $\mathscr{F}$ to be the set of *semi-infinite* words $w = \cdots w_3w_2w_1$ with letters in $\mathbf{B}$ such that

(1) $w_{i+1} \leq w_i$ for $i \geq 1$,
(2) $w_{i+1} = w_i$ implies $|w_i| = 1$,
(3) there exists $c \in \mathbb{Z}$ such that $w_i = \overline{i + c}$ for all $i \gg 1$.

We call c in (3) the *charge of* w. For example, the charge of $w = \cdots \overline{6}\,\overline{5}\,\overline{4}\,\overline{3}\,\overline{1}\,2\,3\,4\,4 \in \mathscr{F}$ is -3. For each $i \in I$, we define the Kashiwara operators

$$e_i, f_i : \mathscr{F} \longrightarrow \mathscr{F} \cup \{0\}$$

as in the case of $\mathcal{W}_{m|n}$ (see Section 2.2). They are well-defined since for $w = \cdots w_3w_2w_1 \in \mathscr{F}$ and $i \in I$,

$$\epsilon^{(i)}(w) = (\cdots , \epsilon^{(i)}(w_3), \epsilon^{(i)}(w_2), \epsilon^{(i)}(w_1))$$

has only finitely many $\pm$'s (we read the signs from left to right). For $w \in \mathscr{F}$, we define

$$\mathrm{wt}(w) = \Lambda + \sum_{b\in\mathbf{B}} m_b\epsilon_b \in P,$$

where $m_b = \left|\{ k \,|\, w_k = b \ (k \geq 1) \}\right| - \delta_{|b|0}$ for $b \in \mathbf{B}$. Since $m_b = 0$ for almost all $b \in \mathbf{B}$, $\mathrm{wt}(w)$ is well-defined. For $w \in \mathscr{F}$ and $i \in I$, we set

$$\varepsilon_i(w) = \max\{ k \,|\, e_i^k w \neq 0 \}, \quad \varphi_i(w) = \max\{ k \,|\, f_i^k w \neq 0 \}.$$

For $c \in \mathbb{Z}$, let

$$\begin{aligned} H^c &= \cdots \ \overline{c+3}\ \overline{c+2}\ \overline{c+1}, && \text{if } c \geq 0, \\ &= \cdots \overline{3}\ \overline{2}\ \overline{1}\ \underbrace{1\ \cdots\ 1}_{|c|}, && \text{if } c \leq 0. \end{aligned}$$

Note that $\mathrm{wt}(H^0) = \Lambda$.

Proposition 5.1. *$\mathscr{F}$ is a crystal graph for $\mathfrak{g}$, and*

$$\mathscr{F} = \bigoplus_{c \in \mathbb{Z}} \mathbf{B}(c),$$

where $\mathbf{B}(c)$ is the connected component of H^c ($c \in \mathbb{Z}$). Moreover, H^c is the highest weight element in $\mathscr{F}$, that is, $\mathrm{wt}(H^c) \geq \mathrm{wt}(w)$ for $w \in \mathbf{B}(c)$.

Proof. The conditions (b), (c) and (d) in Definition 2.1 are satisfied directly. So, it suffices to check that (a) holds. Given $w = \cdots w_3 w_2 w_1 \in \mathscr{F}$ of charge c and $i \in I$, choose a sufficiently large $M > 0$ such that

(i) $w_k = \overline{c+k}$ for all $k \geq M$,
(ii) $\langle h_i, \mathrm{wt}(w^{>M}) \rangle = 0$, where $w^{>M} = \cdots w_{M+2} w_{M+1} \in \mathscr{F}$.

Put $w^{\leq M} = w_M w_{M-1} \cdots w_1$. Then we may view $w^{\leq M}$ as an element in $\mathcal{W}_{m|n}$, where $m = M + c$ and $n \gg 0$. Also, we have $\varepsilon_i(w) = \varepsilon_i(w^{\leq M})$, $\varphi_i(w) = \varphi_i(w^{\leq M})$, and $\langle h_i, \mathrm{wt}(w^{\leq M}) \rangle = \langle h_i, \mathrm{wt}(w) \rangle$. This implies the condition (a).

Since $w^{\leq M}$ is $\mathfrak{gl}_{m|n}$-equivalent to a semistandard tableau $P(w^{\leq M})$ of a single column in $\mathbf{B}_{m|n}((1^M))$, $w^{\leq M}$ is connected to the highest weight element $H^{(1^M)}_{m|n}$. If $P(w^{\leq M}) = f_{i_1} \cdots f_{i_r} H^{(1^M)}_{m|n}$ for some $r \geq 0$ and $i_k \in I_{m|n}$ ($1 \leq k \leq r$), then we have $w = f_{i_1} \cdots f_{i_r} H^c$. In particular, we have $\mathrm{wt}(w) = \mathrm{wt}(H^c) - \sum_{k=1}^r \alpha_{i_k} \leq \mathrm{wt}(H^c)$. $\square$

Let $x = \{\, x_b \,|\, b \in \mathbf{B} \,\}$. For $\mu \in P$, we define $x^\mu = \prod_{b \in \mathbf{B}} x_b^{m_b}$, where $\mu = k\Lambda + \sum_{b \in \mathbf{B}} m_b \epsilon_b$ ($k \in \mathbb{Z}$). Then the character of $\mathscr{F}$ is given by

$$\mathrm{ch}\mathscr{F} = \sum_{w \in \mathscr{F}} x^{\mathrm{wt}(w)} = \frac{\prod_{b \in \mathbf{B}^+} (1 + x_b^{-1})}{\prod_{b' \in \mathbf{B}^-} (1 - x_{b'})}. \tag{5.1}$$

5.2. **Semi-infinite semistandard tableaux for $\mathfrak{g}$.** Let us describe a crystal graph for $\mathfrak{g}$ occurring as a connected component in $\mathscr{F}^{\otimes u}$ ($u \geq 1$). Let

$$\mathbb{Z}^u_+ = \{\, \lambda = (\lambda_1, \cdots, \lambda_u) \in \mathbb{Z}^u \,|\, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_u \,\}$$

be the set of all *generalized partitions* of length u. For each $\lambda = (\lambda_1, \cdots, \lambda_u) \in \mathbb{Z}^u_+$, we call a u-tuple of semi-infinite words $\mathbf{w} = (w^{(1)}, \cdots, w^{(u)}) \in \mathscr{F}^u$ a *semistandard tableau of charge* λ if

(1) $w^{(i)} = \cdots w_3^{(i)} w_2^{(i)} w_1^{(i)} \in \mathscr{F}$, where the charge of $w^{(i)}$ is λ_i for $1 \leq i \leq u$,
(2) $w_k^{(i)} \geq w_{k+d_i}^{(i+1)}$ for $1 \leq i < u$ and $k \geq 1$, where $d_i = \lambda_i - \lambda_{i+1}$,
(3) $w_k^{(i)} = w_{k+d_i}^{(i+1)}$ implies $|w_k^{(i)}| = 0$.

We denote by $\mathbf{B}(\lambda)$ the set of all semistandard tableaux of charge λ. In fact, each $\mathbf{w} \in \mathbf{B}(\lambda)$ determines a unique semi-infinite tableau with infinitely many rows and u columns, where each row of $\mathbf{w}$ reads (from left to right) as follows;

$$w^{(u)}_{k+d_1+\cdots+d_{u-1}} \cdots w^{(3)}_{k+d_1+d_2} w^{(2)}_{k+d_1} w^{(1)}_k,$$

for $k \in \mathbb{Z}$ (we assume that $w^{(i)}_k$ is empty for $k \leq 0$).

Example 5.2. Let $\lambda = (3, 1, -2, -2)$, and let $\mathbf{w} = (w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)})$ be given by

$$\begin{array}{cccc}
w^{(4)} & w^{(3)} & w^{(2)} & w^{(1)} \\
\vdots & \vdots & \vdots & \vdots \\
\overline{6} & \overline{6} & \overline{6} & \overline{6} \\
\overline{5} & \overline{5} & \overline{5} & \overline{5} \\
\overline{4} & \overline{4} & \overline{3} & \overline{1} \\
\overline{3} & \overline{3} & \overline{2} & \\
\overline{2} & \overline{2} & \overline{1} & \\
\overline{1} & \overline{1} & & \\
1 & 2 & & \\
1 & 3 & &
\end{array}.$$

Then $\mathbf{w}$ is a semistandard tableau of charge λ.

For $\lambda \in \mathbb{Z}^u_+$ and $\mathbf{w} = (w^{(1)}, \cdots, w^{(u)}) \in \mathbf{B}(\lambda)$, we may view $\mathbf{w} = w^{(1)} \otimes \cdots \otimes w^{(u)} \in \mathscr{F}^{\otimes u}$, and consider $\mathbf{B}(\lambda)$ as a subset of $\mathscr{F}^{\otimes u}$.

Proposition 5.3. *For $\lambda \in \mathbb{Z}^u_+$, $\mathbf{B}(\lambda)$ together with 0 is stable under e_i and f_i $(i \in I)$. Hence, $\mathbf{B}(\lambda)$ is a crystal graph for $\mathfrak{g}$. Furthermore, $\mathbf{B}(\lambda)$ is a connected I-colored oriented graph with a unique highest weight element H^λ.*

Proof. To show that $\mathbf{B}(\lambda)$ is a crystal graph for $\mathfrak{g}$, it is enough to check that $e_i\mathbf{w}, f_i\mathbf{w} \in \mathbf{B}(\lambda) \cup \{0\}$ for $\mathbf{w} \in \mathbf{B}(\lambda)$ and $i \in I$.

Suppose that $\mathbf{w} \in \mathbf{B}(\lambda)$ is given. Choose sufficiently large $m, n > 0$. Then, for each $1 \leq i \leq u$, we have

$$w^{(i)} = \cdots \overline{m+3}\ \overline{m+2}\ \overline{m+1}\ w^{(i)}_* = H^m w^{(i)}_*,$$

for some $w^{(i)}_* \in \mathcal{W}_{m|n}$. Set $\mathbf{w}_* = w^{(1)}_* \cdots w^{(u)}_* \in \mathcal{W}_{m|n}$. Then, $\mathbf{w}$ is $\mathfrak{gl}_{m|n}$-equivalent to $\mathbf{w}_*$. Note that $P(\mathbf{w}_*)$, the P-tableau of $\mathbf{w}_*$, is nothing but the (m,n)-hook semistandard tableau which is obtained from $\mathbf{w}$ by removing the entries smaller than $\overline{m}$. So, we have $e_i\mathbf{w}, f_i\mathbf{w} \in \mathbf{B}(\lambda) \cup \{0\}$ for $i \in I_{m|n} \subset I$.

Now, consider $H^\lambda = (w^{(1)}, \cdots, w^{(u)}) \in \mathbf{B}(\lambda)$, where for $1 \leq i \leq u$,

$$\begin{aligned}
w^{(i)} &= H^{\lambda_i}, && \text{if } \lambda_i \geq 0, \\
&= H^0 \underbrace{(u-i+1)\cdots(u-i+1)}_{|\lambda_i|}, && \text{if } \lambda_i < 0.
\end{aligned}$$

Since $P(\mathbf{w}_*)$ is connected to the highest weight element, it follows that $\mathbf{w}$ is connected to H^λ. Also, for $\mathbf{w} \in \mathbf{B}(\lambda)$, we have $\mathrm{wt}(w) = \sum_{i=1}^r \mathrm{wt}(w^{(i)})$ and $\mathrm{wt}(\mathbf{w}) \leq \mathrm{wt}(H^\lambda)$. Hence, $\mathbf{B}(\lambda)$ is connected with the unique highest weight element H^λ. $\square$

Lemma 5.4. *For $u \geq 1$, each connected component of $\mathscr{F}^{\otimes u}$ is isomorphic to $\mathbf{B}(\lambda)$ for some $\lambda \in \mathbb{Z}_+^u$.*

Proof. Suppose that $\mathbf{w} = w^{(1)} \otimes \cdots \otimes w^{(u)} \in \mathscr{F}^{\otimes u}$ is given. Choose sufficiently large $m, n > 0$. Then, for each $1 \leq i \leq u$, we have $w^{(i)} = H^m w_*^{(i)}$, for some $w_*^{(i)} \in \mathcal{W}_{m|n}$.

Set $\mathbf{w}_* = w_*^{(1)} \cdots w_*^{(u)} \in \mathcal{W}_{m|n}$. Then $\mathbf{w}$ is $\mathfrak{gl}_{m|n}$-equivalent to $\mathbf{w}_*$, and $P(\mathbf{w}_*) \in \mathbf{B}_{m|n}(\lambda)$ for some $\lambda = (\lambda_k)_{k \geq 1} \in \mathcal{P}_{m|n}$. We see that $\lambda_1 \leq u$ from Schensted's algorithm.

For $1 \leq i \leq u$, let $w_\sharp^{(i)}$ be the word obtained by reading the $(u-i+1)^{\text{th}}$-column of $P(\mathbf{w}_*)$ from top to bottom (note that the left-most column in λ is the first one). Set $\widetilde{\mathbf{w}} = \widetilde{w}^{(1)} \otimes \cdots \otimes \widetilde{w}^{(u)} \in \mathscr{F}^{\otimes u}$ where $\widetilde{w}^{(i)} = H^m w_\sharp^{(i)}$. Then $\widetilde{\mathbf{w}}$ is $\mathfrak{gl}_{m|n}$-equivalent to $\mathbf{w}$. Also, $\widetilde{\mathbf{w}}$ is uniquely determined independent of all sufficiently large m, n, and hence it is $\mathfrak{g}$-equivalent to $\mathbf{w}$.

Let μ_i be the charge of $\widetilde{w}^{(i)}$ $(1 \leq i \leq u)$. Since the length of $w_\sharp^{(i)}$ is less than or equal to that of $w_\sharp^{(i+1)}$, we have $\mu_i \geq \mu_{i+1}$ for $1 \leq i \leq u-1$, and hence $\widetilde{\mathbf{w}}$ is a semistandard tableau of charge μ, where $\mu = (\mu_1, \cdots, \mu_u) \in \mathbb{Z}_+^u$. □

By Lemma 2.9, we can also check the following lemma, which implies that each $\mathbf{w} \in \mathscr{F}^{\otimes u}$ is $\mathfrak{g}$-equivalent to a unique semi-infinite semistandard tableau.

Lemma 5.5. *Let $\mathbf{w}$ and $\mathbf{w}'$ be two semi-infinite semistandard tableaux. If $\mathbf{w} \simeq_{\mathfrak{g}} \mathbf{w}'$, then $\mathbf{w} = \mathbf{w}'$.* □

5.3. **Rational semistandard tableaux for $\mathfrak{gl}_u$.** Let us recall the crystal graphs of rational representations of $\mathfrak{gl}_{u|0}$ for $u \geq 2$. By convention, we write $\mathfrak{gl}_u = \mathfrak{gl}_{u|0}$, $\mathbf{B}_u = \mathbf{B}_{u|0}$, $I_u = I_{u|0}$, $P_u = P_{u|0}$ and so on.

Let $\mathbf{B}_u^\vee = \{ -\overline{1} < -\overline{2} < \cdots < -\overline{u} \}$ be the dual crystal graph of $\mathbf{B}_u$ whose associated graph is given by

$$-\overline{1} \xrightarrow{\ \overline{1}\ } -\overline{2} \xrightarrow{\ \overline{2}\ } \cdots \xrightarrow{\ \overline{u-1}\ } -\overline{u},$$

where $\mathrm{wt}(-\overline{k}) = -\mathrm{wt}(\overline{k}) = -\epsilon_{\overline{k}}$ for $1 \leq k \leq u$ (cf.[13]).

Given $\lambda = (\lambda_1, \cdots, \lambda_u) \in \mathbb{Z}_+^u$, we may identify λ with a *generalized Young diagram* in the following way. First, we fix a vertical line. Then for each λ_k, we place $|\lambda_k|$ nodes (or boxes) in the k^{th} row in a left-justified (resp. right-justified) way with respect to the vertical line if $\lambda_k \geq 0$ (resp. $\lambda_k \leq 0$). For example,

$$\lambda = (3,2,0,-1,-2) \longleftrightarrow \begin{array}{cc|ccc} & & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \\ & & & & \\ & \bullet & & & \\ \bullet & \bullet & & & \\ \text{-2} & \text{-1} & 1 & 2 & 3 \end{array} .$$

We enumerate the columns of a diagram as in the above figure.

Definition 5.6 (cf.[17])**.** Let T be a tableau obtained by filling a generalized Young diagram λ of length u with the entries in $\mathbf{B}_u \cup \mathbf{B}_u^\vee$. We call T a *rational semistandard of shape* λ if

(1) the entries in the columns indexed by positive (resp. negative) numbers belong to $\mathbf{B}_u$ (resp. $\mathbf{B}_u^\vee$),

(2) the entries in each row (resp. column) are weakly (resp. strictly) increasing from left to right (resp. from top to bottom),

(3) if $b_1 < \cdots < b_s$ (resp. $-b'_1 < \cdots < -b'_t$) are the entries in the $1^{\rm st}$ (resp. $-1^{\rm st}$) column ($s+t \leq u$), then
$$b''_i \leq b_i,$$
for $1 \leq i \leq s$, where $\{b''_1 < \cdots < b''_{u-t}\} = \mathbf{B}_u \setminus \{b'_1, \cdots, b'_t\}$.

We denote by $\mathbf{B}_u(\lambda)$ the set of all rational semistandard tableaux of shape λ.

Example 5.7. For $\lambda = (3, 2, 0, -1, -2)$, we have
$$\begin{array}{rr|lll} & & \overline{4} & \overline{3} & \overline{1} \\ & & \overline{2} & \overline{2} & \\ & & & & \\ & -\overline{1} & & & \\ -\overline{2} & -\overline{2} & & & \end{array} \in \mathbf{B}_5(\lambda), \ \text{ but } \quad \begin{array}{rr|lll} & & \overline{4} & \overline{3} & \overline{1} \\ & & \overline{2} & \overline{2} & \\ & & & & \\ & -\overline{4} & & & \\ -\overline{2} & -\overline{5} & & & \end{array} \not\in \mathbf{B}_5(\lambda).$$

Let us explain the relation between the crystal graphs of rational representations and polynomial representations. Let T be a rational semistandard tableau in $\mathbf{B}_u((-1)^t)$ ($0 \leq t \leq u$) with the entries $-b_1 < \cdots < -b_t$. We define $\sigma(T)$ be the tableau in $\mathbf{B}_u(1^{u-t})$ with the entries $b'_1 < \cdots < b'_{u-s}$, where $\{b'_1 < \cdots < b'_{u-t}\} = \mathbf{B}_u \setminus \{b_1 < \cdots < b_t\}$. If $t = u$, then we define $\sigma(T)$ to be the empty word.

Generally, for $\lambda \in \mathbb{Z}^u_+$ and $T \in \mathbf{B}_u(\lambda)$, we define $\sigma(T)$ to be the tableau obtained by applying σ to the $-1^{\rm st}$ column of T. For example, when $u = 5$, we have
$$\sigma\left(\begin{array}{rr|lll} & & \overline{4} & \overline{3} & \overline{1} \\ & & \overline{2} & \overline{2} & \\ & & & & \\ & -\overline{\mathbf{1}} & & & \\ -\overline{2} & -\overline{\mathbf{4}} & & & \end{array}\right) = \begin{array}{r|llll} & \overline{\mathbf{5}} & \overline{4} & \overline{3} & \overline{1} \\ & \overline{\mathbf{3}} & \overline{2} & \overline{2} & \\ & \overline{\mathbf{2}} & & & \\ & & & & \\ -\overline{2} & & & & \end{array}.$$

By Definition 5.6, we have $\sigma(T) \in \mathbf{B}_u(\lambda + (1^u))$, where we define $\mu + \nu = (\mu_k + \nu_k)_{k \geq 1}$ for two generalized partitions $\mu = (\mu_k)_{k\geq 1}$ and $\nu = (\nu_k)_{\geq 1}$ in $\mathbb{Z}^u_+$.

Now, we embed $\mathbf{B}_u(\lambda)$ into $(\mathbf{B}_u \oplus \mathbf{B}^\vee_u)^{\otimes N}$ ($N = \sum_{k\geq 1} |\lambda_k|$) by column reading of tableaux (cf. Section 2.2), and apply e_i, f_i to $\mathbf{B}_u(\lambda)$ ($i \in I_u$). Then

Lemma 5.8. *For $\lambda \in \mathbb{Z}^u_+$, $\mathbf{B}_u(\lambda)$ is a crystal graph for $\mathfrak{gl}_u$, and the map*
$$\sigma : \mathbf{B}_u(\lambda) \cup \{0\} \to \mathbf{B}_u(\lambda + (1^u)) \cup \{0\},$$
($\sigma(0) = 0$) is a bijection which commutes with e_i, f_i for $i \in I_u$, where $\mathrm{wt}(\sigma(T)) = \mathrm{wt}(T) + (\epsilon_{\overline{u}} + \cdots + \epsilon_{\overline{1}})$ for $T \in \mathbf{B}_u(\lambda)$.

Proof. It follows immediately that σ is a bijection and $\mathrm{wt}(\sigma(T)) = \mathrm{wt}(T) + (\epsilon_{\overline{u}} + \cdots + \epsilon_{\overline{1}})$ for $T \in \mathbf{B}_u(\lambda)$.

So, it remains to show that $\mathbf{B}_u(\lambda)$ is a crystal graph for $\mathfrak{gl}_u$, and σ commutes with the Kashiwara operators. We will use induction on the number of columns indexed by negative numbers.

Suppose that $\lambda = ((-1)^t)$ for some $0 \leq t \leq u$. Then it is straightforward to check that our claim holds.

Next, for a general $\lambda \in \mathbb{Z}^u_+$, any tableau T in $\mathbf{B}_u(\lambda)$ can be viewed as a tensor product of its columns when we apply e_i, f_i. By Definition 2.3 and the argument in

the case of a single column, it follows that $\sigma(x_iT) = x_i\sigma(T)$ for $x = e, f$ and $i \in I_u$. Also, by induction hypothesis, $x_i\sigma(T) \in \mathbf{B}_u(\lambda + (1^u)) \cup \{0\}$, which is equivalent to saying that $x_iT \in \mathbf{B}_u(\lambda) \cup \{0\}$. This completes the induction. $\square$

Note that for $\lambda \in \mathbb{Z}_+^u$, there exists a unique highest weight element H_u^λ in $\mathbf{B}_u(\lambda)$. In fact, $H_u^\lambda = \sigma^{-k}(H_u^{\lambda+(k^u)})$ for all $k \geq 0$ such that $\lambda + (k^u)$ is an ordinary partition.

Let $y = \{\, y_b \,|\, b \in \mathbf{B}_u \,\}$ be the set of variables indexed by $\mathbf{B}_u$. For $\mu = \sum_{b\in\mathbf{B}_u} \mu_b\epsilon_b \in P_u$, we set $y^\mu = \prod_{b\in\mathbf{B}_u} y_b^{\mu_b}$. For $\lambda \in \mathbb{Z}_+^u$, the character of $\mathbf{B}_u(\lambda)$ is given by a *rational Schur function* corresponding to λ;

$$s_\lambda(y) = \sum_{T\in\mathbf{B}_u(\lambda)} y^{\mathrm{wt}T}.$$

By Lemma 5.8, we have $s_{\lambda+(1^u)}(y) = (y_{\overline{u}} \cdots y_{\overline{1}})s_\lambda(y)$.

5.4. **Decomposition of $\mathscr{F}^{\otimes u}$.** Now, let us decompose $\mathscr{F}^{\otimes u}$ for $u \geq 2$. We start with another description of $\mathscr{F}^{\otimes u}$ in terms of matrices of non-negative integers. Set

$$\begin{aligned}\mathscr{M}^u = \{A = (a_{bb'})_{b\in\mathbf{B},b'\in\mathbf{B}_u} \,|\, & \\ &(1)\ a_{bb'} \in \mathbb{Z}_{\geq 0}, \\ &(2)\ a_{bb'} \leq 1 \text{ if } |b| = 0, \\ &(3)\ a_{bb'} = 1 \text{ for all } b \ll \overline{1}, \text{ and } a_{bb'} = 0 \text{ for all } b \gg 1\,\}.\end{aligned} \tag{5.2}$$

For $\mathbf{w} = w^{(1)} \otimes \cdots \otimes w^{(u)} \in \mathscr{F}^{\otimes u}$, set $A(\mathbf{w}) = (a_{b\,\overline{k}}) \in \mathscr{M}^u$, where $a_{b\,\overline{k}}$ is the number of occurrences of b in $w^{(u-k+1)}$ $(1 \leq k \leq u)$. Then the map $\mathbf{w} \mapsto A(\mathbf{w})$ is a bijection from $\mathscr{F}^{\otimes u}$ to $\mathscr{M}^u$, where each $w^{(i)}$ corresponds to the i^{th}-column of $A(\mathbf{w})$ for $1 \leq i \leq u$.

Example 5.9.

$$\mathscr{F}^{\otimes 3} \ni \mathbf{w} = \begin{matrix}\vdots \\ \overline{5} \\ \overline{4} \\ \overline{2} \\ 2 \\ 2 \\ \end{matrix} \otimes \begin{matrix}\vdots \\ \overline{5} \\ \overline{4} \\ \overline{3} \\ 3 \\ \\ \end{matrix} \otimes \begin{matrix}\vdots \\ \overline{5} \\ \overline{3} \\ \overline{2} \\ 1 \\ 1 \\ 1\end{matrix} \quad \longleftrightarrow \quad A(\mathbf{w}) = \left(\begin{array}{ccc}\vdots & \vdots & \vdots \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 3 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots\end{array}\right) \in \mathscr{M}^3.$$

Let us define the crystal graph structures on $\mathscr{M}^u$, which are naturally induced from those on $\mathbf{M}^\sharp_{m|n,u|0}$ in Section 4. For $m, n > 0$, we define

$$\mathrm{Res}_{m|n} : \mathscr{M}^u \longrightarrow \mathbf{M}^\sharp_{m|n,u|0},$$

by $\mathrm{Res}_{m|n}A = (a_{bb'})_{b\in\mathbf{B}_{m|n},b'\in\mathbf{B}_u}$ for $A = (a_{bb'}) \in \mathscr{M}^u$.

For $A \in \mathscr{M}^u$ and $i \in I$, we define $e_i A$ and $f_i A$ to be the unique elements in $\mathscr{M}^u \cup \{0\}$ satisfying

$$\mathrm{Res}_{m|n}(e_i A) = e_i(\mathrm{Res}_{m|n} A), \quad \mathrm{Res}_{m|n}(f_i A) = f_i(\mathrm{Res}_{m|n} A),$$

for all sufficiently large $m, n > 0$, where we assume that $\mathrm{Res}_{m|n} 0 = 0$. We set

$$\begin{aligned} \mathrm{wt}(A) &= u\Lambda + \sum_{b \in \mathbf{B}} m_b \epsilon_b, \\ \varepsilon_i(A) &= \max\{\, k \,|\, e_i^k A \neq 0 \,\}, \quad \varphi_i(A) = \max\{\, k \,|\, f_i^k A \neq 0 \,\}, \end{aligned}$$

for $A \in \mathscr{M}^u$ and $i \in I$, where $m_b = \sum_{b' \in \mathbf{B}_u} (a_{bb'} - \delta_{|b|0})$. Then it is not difficult to check the following lemma;

Lemma 5.10. *$\mathscr{M}^u$ is a crystal graph for $\mathfrak{g}$, and the map $\mathbf{w} \mapsto A(\mathbf{w})$ is an isomorphism of $\mathfrak{g}$-crystals from $\mathscr{F}^{\otimes u}$ to $\mathscr{M}^u$. In particular, for $A \in \mathscr{M}^u$, A is $\mathfrak{g}$-equivalent to a unique semistandard tableau in $\mathbf{B}(\lambda)$ for some $\lambda \in \mathbb{Z}_+^u$.* □

Next, let us define a $\mathfrak{gl}_u$-crystal structure on $\mathscr{M}^u$. Since $\mathscr{M}^u$ can be identified with $\mathscr{F}^{\otimes u}$, this induces a $\mathfrak{gl}_u$-crystal structure on $\mathscr{F}^{\otimes u}$. For $A \in \mathscr{M}^u$ and $j \in I_u$, we define $e_j^* A$ and $f_j^* A$ to be the unique elements in $\mathscr{M}^u \cup \{0\}$ satisfying

$$\mathrm{Res}_{m|n}(e_j^* A) = e_j^*(\mathrm{Res}_{m|n} A), \quad \mathrm{Res}_{m|n}(f_j^* A) = f_j^*(\mathrm{Res}_{m|n} A),$$

for all sufficiently large $m, n > 0$. We set

$$\begin{aligned} \mathrm{wt}^*(A) &= \sum_{b \in \mathbf{B}_u} m_b \epsilon_b, \\ \varepsilon_j^*(A) &= \max\{\, k \,|\, (e_j^*)^k A \neq 0 \,\}, \quad \varphi_j^*(A) = \max\{\, k \,|\, (f_j^*)^k A \neq 0 \,\}, \end{aligned}$$

for $A \in \mathscr{M}^u$ and $j \in I_u$, where $m_b = \sum_{b' \in \mathbf{B}} (a_{b'b} - \delta_{|b'|0})$. Then

Lemma 5.11. *$\mathscr{M}^u$ is a crystal graph for $\mathfrak{gl}_u$ with respect to e_j^*, f_j^* ($j \in I_u$), and each $A \in \mathscr{M}^u$ is $\mathfrak{gl}_u$-equivalent to a unique rational semistandard tableau.*

Proof. It is clear that $\mathscr{M}^u$ is a crystal graph for $\mathfrak{gl}_u$. For $A \in \mathscr{M}^u$, choose sufficiently large m, n. As an element in a $\mathfrak{gl}_u$-crystal, consider a semistandard tableau $T = P_2^\sharp(\mathrm{Res}_{m|n} A)$ (see (4.6)). We observe that $\mathrm{wt}(T) = \mathrm{wt}^*(A) + k(\epsilon_{\overline{u}} + \cdots + \epsilon_{\overline{1}})$ for some $k \geq 0$. Then, by Lemma 5.8, the rational semistandard tableau $\sigma^{-k}(T)$, say S, is $\mathfrak{gl}_u$-equivalent to A. Note that S is independent of m and n since $\mathrm{wt}^*(A)$ is fixed. □

Suppose that $A \in \mathscr{M}^u$ is given. By Lemma 5.10, there exists a unique semistandard tableau $\mathbf{w}$ in $\mathbf{B}(\lambda)$ for some $\lambda \in \mathbb{Z}_+^u$, which is $\mathfrak{g}$-equivalent to A, and we set $\mathscr{P}_1(A) = \mathbf{w}$. Also, by Lemma 5.11, there exists a unique rational semistandard tableau T, which is $\mathfrak{gl}_u$-equivalent to A, and we set $\mathscr{P}_2(A) = T$. Now, we define

$$\varpi(A) = (\mathscr{P}_1(A), \mathscr{P}_2(A)). \tag{5.3}$$

Proposition 5.12. *$\mathscr{M}^u$ is a $(\mathfrak{g}, \mathfrak{gl}_u)$-bicrystal, and for each connected component $\mathscr{C}$ in $\mathscr{M}^u$, ϖ gives the following isomorphism of $(\mathfrak{g}, \mathfrak{gl}_u)$-bicrystals;*

$$\varpi : \mathscr{C} \longrightarrow \mathbf{B}(\lambda) \times \mathbf{B}_u(\mu),$$

for some $\lambda, \mu \in \mathbb{Z}_+^u$.

Proof. It follows from Proposition 4.2. $\square$

Example 5.13. Let A be given in Example 5.9. Then

$$A \;\simeq_{\mathfrak{g}}\; \begin{matrix} \vdots \\ \overline{5} \\ \overline{4} \\ \overline{2} \\ 2 \\ 2 \end{matrix} \otimes \begin{matrix} \vdots \\ \overline{5} \\ \overline{4} \\ \overline{3} \\ 3 \\ \end{matrix} \otimes \begin{matrix} \vdots \\ \overline{5} \\ \overline{3} \\ \overline{2} \\ 1 \\ 1 \\ 1 \end{matrix} \;\simeq_{\mathfrak{g}}\; \begin{matrix} \vdots & \vdots & \vdots \\ \overline{5} & \overline{5} & \overline{5} \\ \overline{4} & \overline{4} & \overline{2} \\ \overline{3} & \overline{3} & \\ \overline{2} & 2 & \\ 1 & 2 & \\ 1 & 3 & \\ 1 & & \end{matrix} \;=\; \mathscr{P}_1(A) \in \mathbf{B}(3,-1,-2),$$

On the other hand, consider $\mathrm{Res}_{4|3}A$. Then

$$\begin{aligned} \mathrm{Res}_{4|3}A \;&\simeq_{\mathfrak{gl}_u}\; \overline{2} \otimes \overline{3}\,\overline{3} \otimes \overline{1}\,\overline{1}\,\overline{1} \otimes \begin{matrix} \overline{3} \\ \overline{1} \end{matrix} \otimes \begin{matrix} \overline{2} \\ \overline{1} \end{matrix} \otimes \begin{matrix} \overline{3} \\ \overline{2} \end{matrix} \\ &\simeq_{\mathfrak{gl}_u}\; \begin{matrix} \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{2} & \overline{1} \\ \overline{2} & \overline{2} & \overline{1} & \overline{1} & \overline{1} & \\ \overline{1} & & & & & \end{matrix} \;=\; P_2^{\sharp}(\mathrm{Res}_{4|3}A). \end{aligned}$$

Since $\mathrm{wt}(A) = -\epsilon_{\overline{2}} + \epsilon_{\overline{1}}$, we have

$$\mathscr{P}_2(A) = \sigma^{-4}(\; P_2^{\sharp}(\mathrm{Res}_{4|3}A)) \;=\; \left.\begin{matrix} & & \\ -\overline{1} & -\overline{2} & -\overline{2} \end{matrix}\;\right|\; \begin{matrix} \overline{2} & \overline{1} \\ \overline{1} & \\ & \end{matrix} \quad \in \mathbf{B}_3(2,1,-3).$$

Now, we let

$$\mathscr{M}^u_{\text{h.w.}} = \{\, A \,|\, \varpi(A) = (H^{\lambda}, H^{\mu}_u) \;\text{ for some } \lambda, \mu \in \mathbb{Z}^u_+ \;\} \tag{5.4}$$

be the set of all the highest weight elements in $\mathscr{M}^u$. Then $\mathscr{M}^u$ is the direct sum of the connected components of the elements in $\mathscr{M}^u_{\text{h.w.}}$ as a $(\mathfrak{g}, \mathfrak{gl}_u)$-bicrystal. For $\lambda \in \mathbb{Z}^u_+$, let μ_i $(1 \le i \le u)$ be the number of occurrences of i in H^{λ}. Put $\nu_i = \mu_i - \mu_{i+1}$ for $1 \le i \le u$ where $\mu_{u+1} = 0$. Then we define $\mathscr{A}_{\lambda} = (a_{bb'}) \in \mathscr{M}^u$ by

(1) for $k \ge 1$, $0 \le l < u$,

$$a_{\overline{k}\,\overline{u-l}} = \begin{cases} 1, & \text{if there exists } \overline{k} \text{ in the } (l+1)^{\text{th}}\text{-column of } H^{\lambda}, \\ 0, & \text{otherwise.} \end{cases}$$

(2) for $k \ge 1$, $0 \le l < u$,

$$a_{k\,\overline{u-l}} = \begin{cases} \nu_{k+l}, & \text{if } 1 \le k+l \le u, \\ 0, & \text{otherwise,} \end{cases}$$

(cf.(4.8)). Then, we can check that $\varpi(\mathscr{A}_{\lambda}) = (H^{\lambda}, H^{\lambda^*}_u)$, where

$$\lambda^* = (-\lambda_u, \cdots, -\lambda_1) \in \mathbb{Z}^u_+.$$

Theorem 5.14. *We have*

$$\mathscr{M}^u_{\text{h.w.}} = \{\, \mathscr{A}_{\lambda} \,|\, \lambda \in \mathbb{Z}^u_+ \,\},$$

and the following isomorphism of $(\mathfrak{g}, \mathfrak{gl}_u)$*-bicrystals;*

$$\varpi : \mathscr{M}^u \longrightarrow \bigoplus_{\lambda \in \mathbb{Z}^u_+} \mathbf{B}(\lambda) \times \mathbf{B}_u(\lambda^*).$$

Proof. Let A be a highest weight element in $\mathscr{M}^u$. Then, $\mathrm{Res}_{m|n} A$ is a highest weight element in $\mathbf{M}^\sharp_{m|n,u|0}$ for all sufficiently large $m, n > 0$. By Theorem 4.5, it follows that $A = \mathscr{A}_\lambda$ for some $\lambda \in \mathbb{Z}^u_+$. □

The character of $\mathscr{M}^u$ is given by

$$\mathrm{ch}\mathscr{M}^u = \sum_{A \in \mathscr{M}^u} x^{\mathrm{wt}(A)} y^{\mathrm{wt}^*(A)} = \frac{\prod_{b \in \mathbf{B}^+} \prod_{b' \in \mathbf{B}_u} (1 + x_b^{-1} y_{b'}^{-1})}{\prod_{b \in \mathbf{B}^-} \prod_{b' \in \mathbf{B}_u} (1 - x_b y_{b'})}.$$

By Theorem 5.14, we obtain the following identity;

$$\frac{\prod_{b \in \mathbf{B}^+} \prod_{b' \in \mathbf{B}_u} (1 + x_b^{-1} y_{b'}^{-1})}{\prod_{b \in \mathbf{B}^-} \prod_{b' \in \mathbf{B}_u} (1 - x_b y_{b'})} = \sum_{\lambda \in \mathbb{Z}^u_+} \mathrm{ch}\mathbf{B}(\lambda) s_{\lambda^*}(y), \tag{5.5}$$

where $\mathrm{ch}\mathbf{B}(\lambda) = \sum_{\mathbf{w} \in \mathbf{B}(\lambda)} x^{\mathrm{wt}(\mathbf{w})}$ is the character of $\mathbf{B}(\lambda)$, and $s_{\lambda^*}(y)$ is the rational Schur function corresponding to λ^*.

From the classical Cauchy identities of Schur functions (cf.[15]), it follows that the left hand side of (5.5) is equal to

$$\sum_{\mu, \nu \in \mathcal{P}} s_\mu(y^{-1}) s_{\mu'}(x_+^{-1}) s_\nu(y) s_\nu(x_-),$$

where $\mathcal{P}$ is the set of all partitions, $y^{-1} = \{\, y_b^{-1} \,|\, b \in \mathbf{B}_u \,\}$, $x_+^{-1} = \{\, x_b^{-1} \,|\, b \in \mathbf{B}^+ \,\}$, and $x_- = \{\, x_b \,|\, b \in \mathbf{B}^- \,\}$. Note that $s_\mu(y^{-1}) = s_{\mu^*}(y)$ and

$$s_{\mu^*}(y) s_\nu(y) = \sum_{\lambda \in \mathbb{Z}^u_+} N^\lambda_{\mu\nu} s_\lambda(y),$$

for some $N^\lambda_{\mu\nu} \in \mathbb{Z}_{\geq 0}$. Then the left hand side of (5.5) can be written as

$$\sum_{\lambda \in \mathbb{Z}^u_+} \left(\sum_{\mu, \nu \in \mathcal{P}} N^\lambda_{\mu\nu} s_{\mu'}(x_+^{-1}) s_\nu(x_-) \right) s_\lambda(y).$$

From a linear independence of $\{\, s_\nu(y) \,|\, \nu \in \mathbb{Z}^u_+ \,\}$ (see Lemma 3.1 [3]), we obtain the following character formula of $\mathbf{B}(\lambda)$ by comparing with the right hand side of (5.5);

Corollary 5.15 (cf.[3, 10])**.** *For* $\lambda \in \mathbb{Z}^u_+$*, we have*

$$\mathrm{ch}\mathbf{B}(\lambda) = \sum_{\mu, \nu \in \mathcal{P}} N^{\lambda^*}_{\mu\nu} s_{\mu'}(x_+^{-1}) s_\nu(x_-).$$

□

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